# Number Theory for Cryptography

## 密碼學與應用

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# Congruence

- **Modulo Operation:**
  - **Question:** What is 12 mod 9?
  - **Answer:** 12 mod 9 ≡ 3 or 12 ≡ 3 (mod 9)

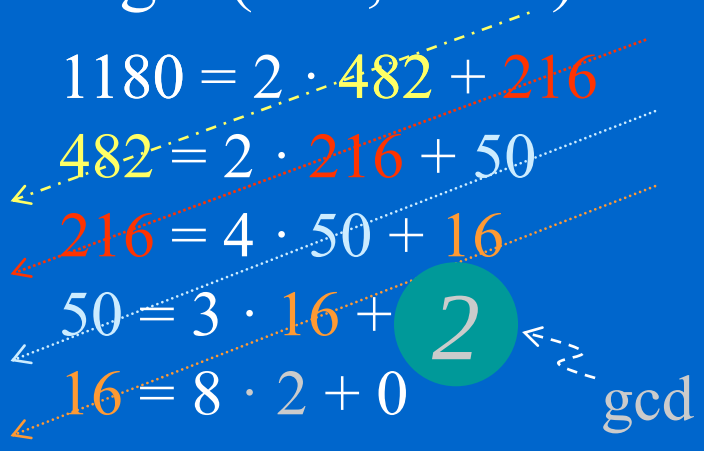
    "12 is congruent to 3 modulo 9"
- **Definition:** Let $a, r, m \in Z$ (where Z is the set of all integers) and $m > 0$. We write
  - $a \equiv r \pmod{m}$ if $m$ divides $a - r$ (i.e. m | $a$-$r$)
  - $m$ is called the *modulus*
  - $r$ is called the *remainder*
  - $a = q \cdot m + r \qquad 0 \leq r < m$
- **Example:** $a = 42$ and $m=9$
  - $42 = 4 \cdot 9 + 6$ therefore $42 \equiv 6 \pmod{9}$

# Greatest Common Divisor

- GCD of a and b is the largest positive integer dividing both a and b
- gcd(a, b) or (a,b)
- ex. gcd(6, 4) = 2, gcd(5, 7) = 1
- Euclidean algorithm

  remainder→divisor → dividend → ignore

  - ex. gcd(482, 1180)




Why does it work?

Let d = gcd(482, 1180)

d | 482 and d | 1180 ⇒ d | 216

because 216 = 1180 - 2 · 482

d | 216 and d | 482 ⇒ d | 50

d | 50 and d | 216 ⇒ d | 16

d | 16 and d | 50 ⇒ d | 2

2 | 16 ⇒ d = 2

# Greatest Common Divisor (cont'd)

- Euclidean Algorithm: calculating GCD

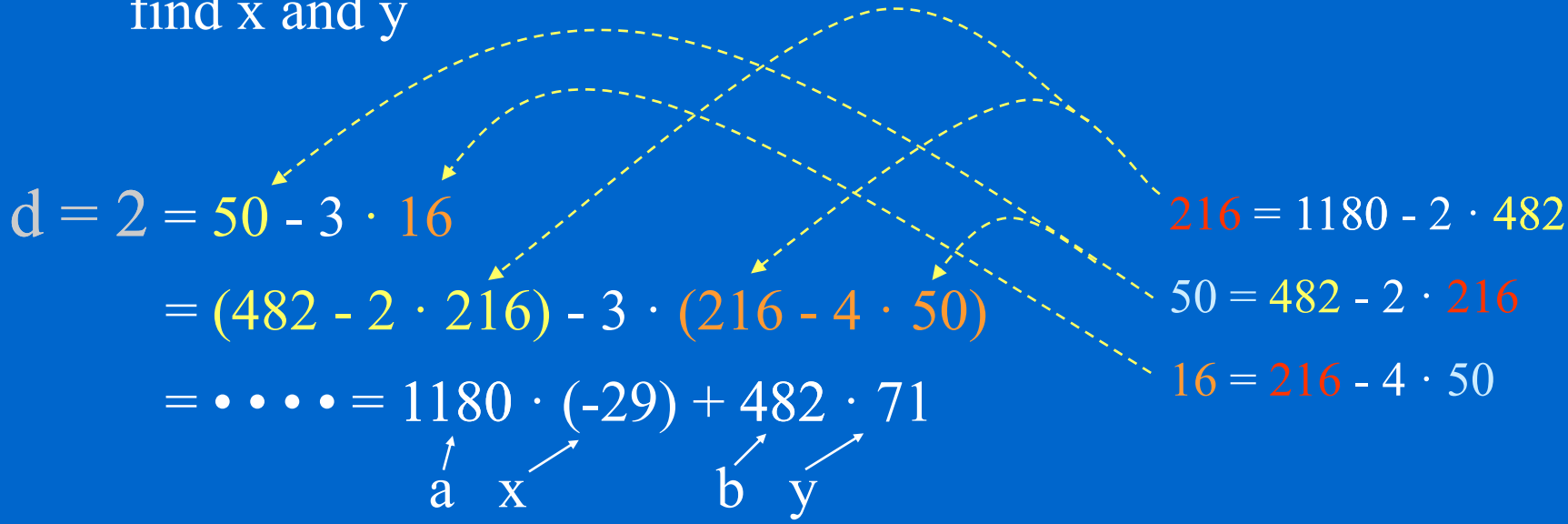
gcd(1180, 482)

(輾轉相除法)

| | | | |
|---|---|---|---|
| 2 | 482<br>432 | 1180<br>964 | 2 |
| 3 | 50<br>48 | 216<br>200 | 4 |
| | 2 | 16<br>16 | 8 |
| | | 0 | |

# Greatest Common Divisor (cont'd)

- Def: a and b are relatively prime: gcd(a, b) = 1
- Theorem: Let a and b be two integers, with at least one of a, b nonzero, and let d = gcd(a,b). Then there exist integers x, y, gcd(x, y) = 1 such that $a \cdot x + b \cdot y = d$
  - Constructive proof: Using Extended Euclidean Algorithm to find x and y



# Extended Euclidean Algorithm

Let gcd(a, b) = d

- Looking for s and t, gcd(s, t) = 1 s.t. $a \cdot s + b \cdot t = d$
- When d = 1, $t \equiv b^{-1} \pmod a$

$a = q_1 \cdot b + r_1$ ①

$b = q_2 \cdot r_1 + r_2$ ②③

$r_1 = q_3 \cdot r_2 + r_3$ ④⑤

$r_2 = q_4 \cdot r_3 + d$

$r_3 = q_5 \cdot d + 0$

Ex. $1180 = 2 \cdot 482 + 216$

$1180 - 2 \cdot 482 = 216$

$482 = 2 \cdot 216 + 50$

$482 - 2 \cdot (1180 - 2 \cdot 482) = 50$

$-2 \cdot 1180 + 5 \cdot 482 = 50$

$216 = 4 \cdot 50 + 16$

$(1180 - 2 \cdot 482) - 4 \cdot (-2 \cdot 1180 + 5 \cdot 482) = 16$

$9 \cdot 1180 - 22 \cdot 482 = 16$

$50 = 3 \cdot 16 + \mathbf{2}$

$(-2 \cdot 1180 + 5 \cdot 482) - 3 \cdot (9 \cdot 1180 - 22 \cdot 482) = 2$

$-29 \cdot 1180 + 71 \cdot 482 = 2$

# Greatest Common Divisor (cont'd)

- The above proves only the existence of integers x and y
- How about gcd(x, y)?

$$d = a \cdot x + b \cdot y \quad \in Z$$
$$d = \gcd(a, b) \quad \Rightarrow \quad 1 = a/d \cdot x + b/d \cdot y$$

If $\gcd(x, y) = r$ then $1 = a/d \cdot (x' \cdot r) + b/d \cdot (y' \cdot r)$

i.e. $1 = r \cdot (a/d \cdot x' + b/d \cdot y')$

which means that $r \mid 1$ i.e. $r = 1$

$\gcd(x, y) = 1$ ¶

Note: gcd(x, y) = 1 but (x, y) is not unique

e.g. $d = a\,x + b\,y = a\,(x - kb) + b\,(y + ka)$

# Greatest Common Divisor (cont'd)

Lemma: $\gcd(a,b) = \gcd(x,y) = \gcd(a,y) = \gcd(x,b) = 1 \Leftrightarrow$

$\exists\ a, b, x, y$ s.t. $1 = a\,x + b\,y$

pf:($\Rightarrow$)

following the previous theorem

($\Leftarrow$)

Given a, b, z, if $\exists$ x, y, $\gcd(x,y)=1$ s.t. $z = ax + by$
then $\gcd(a, b) \mid z$ (also $\gcd(a, y) \mid z$, $\gcd(x, b) \mid z$)

(let $d = \gcd(a, b) \Rightarrow d \mid a$ and $d \mid b \Rightarrow d \mid a\,x + b\,y \Rightarrow d \mid z$)

especially, given a, b, $\exists$ x, y s.t. $1 = a\,x + b\,y$

$\Rightarrow \gcd(a, b) \mid 1 \quad \Rightarrow \gcd(a, b) = 1$

# Operations under mod n

- Proposition:

  Let a,b,c,d,n be integers with $n \neq 0$, suppose

  $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$ then

  $a + c \equiv b + d \pmod{n}$,

  $a - c \equiv b - d \pmod{n}$,

  $a \cdot c \equiv b \cdot d \pmod{n}$

- Proposition:

  Let a,b,c,n be integers with $n \neq 0$ and $\gcd(a,n) = 1$.

  If $a \cdot b \equiv a \cdot c \pmod{n}$ then $b \equiv c \pmod{n}$

# Operations under mod n

- What is the multiplicative inverse of a (mod n)?

  i.e. $a \cdot a^{-1} \equiv 1 \pmod{n}$ or $a \cdot a^{-1} = 1 + k \cdot n$

  $\gcd(a, n) = 1 \Rightarrow \exists\ s$ and t such that $a \cdot s + n \cdot t = 1$

  $\Rightarrow a^{-1} \equiv s \pmod{n}$

This expression also implies gcd(a,n)=1.

- $a \cdot x \equiv b \pmod{n}$, $\gcd(a, n) = 1$, $x \equiv ?$

  $x \equiv b \cdot a^{-1} \equiv b \cdot s \pmod{n}$

- $a \cdot x \equiv b \pmod{n}$, $\gcd(a, n) = d > 1$, $x \equiv ?$

Are there any solutions?

  if $d \mid b$ $(a/d) \cdot x \equiv (b/d) \pmod{n/d}$ $\gcd(a/d, n/d) = 1$

  $x_0 \equiv (b/d) \cdot (a/d)^{-1} \pmod{n/d}$

  $\Rightarrow$ there are d solutions to the equation $a \cdot x \equiv b \pmod{n}$:

  $x_0, x_0+(n/d), \ldots, x_0+(d-1)\cdot(n/d) \pmod{n}$

# Matrix inversion under mod n

- A square matrix is invertible mod n if and only if its determinant and n are relatively prime
- ex: in real field R

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

In a finite field Z (mod n)? we need to find the inverse for ad-bc (mod n) in order to calculate the inverse of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \equiv (ad - bc)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \pmod{n}$$

# Group

- A group G is a finite or infinite set of elements and a binary operation × which together satisfy

| | | | |
|---|---|---|---|
| 1. Closure: | $\forall\, a,b \in G$ | $a \times b = c \in G$ | 封閉性 |
| 2. Associativity: | $\forall\, a,b,c \in G$ | $(a \times b) \times c = a \times (b \times c)$ | 結合性 |
| 3. Identity: | $\forall\, a \in G$ | $1 \times a = a \times 1 = a$ | 單位元素 |
| 4. Inverse: | $\forall\, a \in G$ | $a \times a^{-1} = 1 = a^{-1} \times a$ | 反元素 |

- Abelian group 交換群 $\forall\, a,b \in G$ $a \times b = b \times a$

- Cyclic group G of order m: a group defined by an element $g \in G$ such that $g, g^2, g^3, \ldots g^m$ are all distinct elements in G (thus cover all elements of G) and $g^m = 1$, the element g is called a generator of G. Ex: $Z_n^*$ (or $Z/nZ$)

**means $g \times g \times g \times \ldots \times g$**

# Group (cont'd)

- The **order of a group**: the number of elements in a group G, denoted |G|. If the order of a group is a finite number, the group is said to be a finite group, note $g^{|G|} = 1$ (the identity element).
- The **order of an element g** of a finite group G is the smallest power m such that $g^m = 1$ (the identity element), denoted by $ord_G(g)$
- ex: $\mathbf{Z_n}$: additive group modulo n is the set {0, 1, …, n-1}
  - binary operation: + (mod n)
  - identity: 0
  - inverse: -x ≡ n-x (mod n)

  size of $Z_n$ is n,
  $\underbrace{g+g+\ldots+g} \equiv 0 \pmod n$
- ex: $\mathbf{Z_n^*}$: multiplicative group modulo n is the set {i:0<i<n, gcd(i,n)=1}
  - binary operation: × (mod n)
  - identity: 1
  - inverse: $x^{-1}$ can be found using extended Euclidean Algorithm

  size of $Z_n^*$ is $\phi(n)$,
  $g^{\phi(n)} \equiv 1 \pmod n$

# Ring $Z_m$

- **Definition:** The ring $Z_m$ consists of
  - The set $Z_m = \{0, 1, 2, \ldots, m\text{-}1\}$
  - Two operations "+ (mod m)" and "× (mod m)" for all $a, b \in Z_m$ such that they satisfy the properties on the next slide
- **Example:** $m = 9$ $Z_9 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

  $6 + 8 = 14 \equiv 5 \pmod 9$

  $6 \times 8 = 48 \equiv 3 \pmod 9$

# Properties of the ring $Z_m$

- Consider the ring $Z_m = \{0, 1, \ldots, m\text{-}1\}$
  - The additive identity "0": $a + 0 \equiv a \pmod m$
  - The additive inverse of $a$: $-a = m - a$ s.t. $a + (-a) \equiv 0 \pmod m$
  - Addition is closed i.e if $a, b \in Z_m$ then $a + b \in Z_m$
  - Addition is commutative $a + b \equiv b + a \pmod m$
  - Addition is associative $(a + b) + c \equiv a + (b + c) \pmod m$

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  - Multiplicative identity "1": $a \times 1 \equiv a \pmod m$
  - The multiplicative inverse of $a$ exists only when $\gcd(a,m) = 1$ and denoted as $a^{-1}$ s.t. $a^{-1} \times a \equiv 1 \pmod m$ **might or might not exist**
  - Multiplication is closed i.e. if $a, b \in Z_m$ then $a \times b \in Z_m$
  - Multiplication is commutative $a \times b \equiv b \times a \pmod m$
  - Multiplication is associative $(a \times b) \times c \equiv a \times (b \times c) \pmod m$

# Some remarks on the ring $Z_m$

- A ring is an Abelian group under addition and a semigroup under multiplication.
- A semigroup is defined for a set and a binary operator in which the multiplication operation is associative. No other restrictions are placed on a semigroup; thus a semigroup need not have an **identity** element and its elements need not have **inverses** within the semigroup.

# Some remarks on the ring $Z_m$ (cont'd)

- Roughly speaking a ring is a mathematical structure in which we can add, subtract, multiply, and even sometimes divide. (A ring in which every element has multiplicative inverse is called a field.)
  - **Example:** Is the division 4/15 (mod 26) possible?

    In fact, $4/15 \bmod 26 \equiv 4 \times 15^{-1} \pmod{26}$

    Does $15^{-1} \pmod{26}$ exist ?

    It exists only if gcd(15, 26) = 1.

    $15^{-1} \equiv 7 \pmod{26}$ therefore,

    $4/15 \bmod 26 \equiv 4 \times 7 \equiv 28 \equiv 2 \bmod 26$

# Some remarks on the group $Z_m$ *and* $Z_m^*$

- The modulo operation can be applied whenever we want

  under $Z_m$

  $$(a + b) \pmod m \equiv [(a \pmod m) + ((b \bmod m))] \pmod m$$

  under $Z_m^*$

  $$(a \times b) \pmod m \equiv [(a \pmod m) \times ((b \bmod m))] \pmod m$$

  $$a^b \pmod m \equiv (a \pmod m)^b \pmod m$$

  - Question? $a^b \pmod m \stackrel{?}{\equiv} a^{(b \bmod m)} \pmod m$

# Exponentiation in $Z_m$

- Example: $3^8 \pmod 7 \equiv ?$

  $3^8 \pmod 7 \equiv 6561 \pmod 7 \equiv 2$ since $6561 \equiv 937 \times 7 + 2$ or

  $3^8 \pmod 7 \equiv 3^4 \times 3^4 \pmod 7 \equiv 3^2 \times 3^2 \times 3^2 \times 3^2 \pmod 7$

  $\equiv (3^2 \pmod 7) \times (3^2 \pmod 7) \times (3^2 \pmod 7) \times (3^2 \pmod 7)$

  $\equiv 2 \times 2 \times 2 \times 2 \pmod 7 \equiv 16 \pmod 7 \equiv 2$

- The cyclic group $Z_m^*$ and the modulo arithmetic is of central importance to modern public-key cryptography. In practice, the order of the integers involved in PKC are in the range of $[2^{160}, 2^{1024}]$. Perhaps even larger.

# Exponentiation in $Z_m$ (cont'd)

- How do we do the exponentiation efficiently?
- $3^{1234} \pmod{789}$ many ways to do this
  - a. do 1234 times multiplication and then calculate remainder
  - b. repeat 1234 times (multiplication by 3 and calculate remainder)
  - c. repeated $\lfloor \log 1234 \rfloor$ times (square, multiply and calculate remainder)

    ex. first tabulate

$3^2 \equiv 9 \pmod{789}$ $\quad 3^{32} \equiv 459^2 \equiv 18$ $\quad 3^{512} \equiv 732^2 \equiv 93$

$3^4 \equiv 9^2 \equiv 81$ $\quad 3^{64} \equiv 18^2 \equiv 324$ $\quad 3^{1024} \equiv 93^2 \equiv 759$

$3^8 \equiv 81^2 \equiv 249$ $\quad 3^{128} \equiv 324^2 \equiv 39$

$3^{16} \equiv 249^2 \equiv 459$ $\quad 3^{256} \equiv 39^2 \equiv 732$

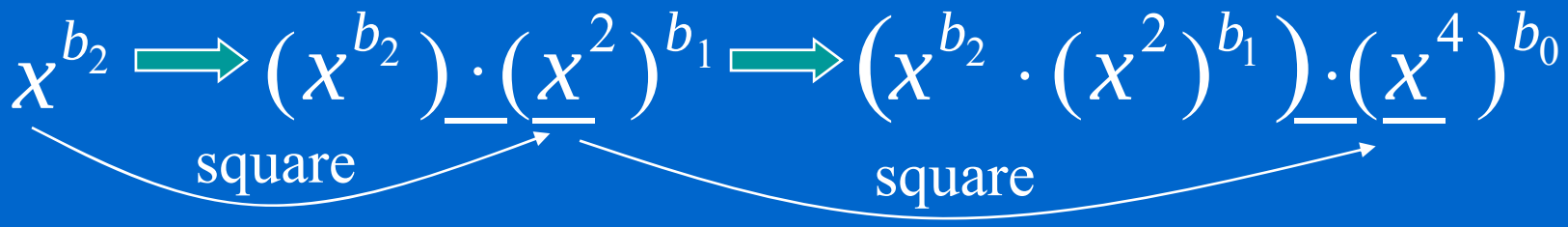
$1234 = 1024 + 128 + 64 + 16 + 2 \qquad (10011010010)_2$

$3^{1234} \equiv 3^{(1024+128+64+16+2)} \equiv (((759 \cdot 39) \cdot 324) \cdot 459) \cdot 9 \equiv 105 \pmod{789}$

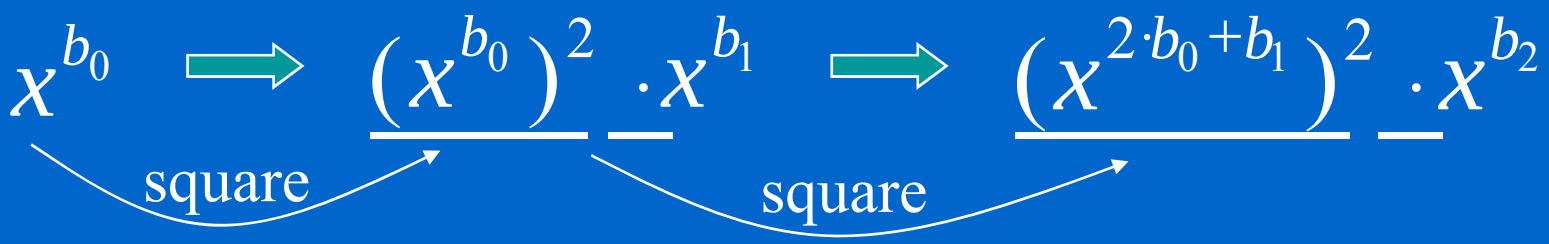
# Exponentiation in $Z_m$ (cont'd)

calculate $x^y \pmod m$ where $y = b_0 \cdot 2^2 + b_1 \cdot 2 + b_2$

- Method 1:




- Method 2:




square and multiply $\lfloor \log y \rfloor$ times

# Exponentiation in $Z_m$ (cont'd)

Method 1:

$1234 = 1024 + 128 + 64 + 16 + 2 \qquad (10011010010)_2$

$3^{1234} \equiv 3^{0+2(1+2(0+2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1))))))))))}$

$\equiv 9 \cdot 9^{2(0+2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1)))))))))}$

$\equiv 9 \cdot 81^{2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1))))))))}$

$\equiv 9 \cdot 249^{2(1+2(0+2(1+2(1+2(0+2(0+2(1))))))))}$

$\equiv 9 \cdot 459 \cdot 459^{2(0+2(1+2(1+2(0+2(0+2(1))))))}$

$\equiv 9 \cdot 459 \cdot 18^{2(1+2(1+2(0+2(0+2(1)))))}$

$\equiv 9 \cdot 459 \cdot 324 \cdot 324^{2(1+2(0+2(0+2(1))))}$

$\equiv 9 \cdot 459 \cdot 324 \cdot 39 \cdot 39^{2(0+2(0+2(1)))}$

$\equiv 9 \cdot 459 \cdot 324 \cdot 39 \cdot 732^{2(0+2(1))}$

$\equiv 9 \cdot 459 \cdot 324 \cdot 39 \cdot 93^{2\,(1)}$

$\equiv 9 \cdot 459 \cdot 324 \cdot 39 \cdot 759 \bmod 789$

# Exponentiation in $Z_m$ (cont'd)

Method 2: $1234 = 1024 + 128 + 64 + 16 + 2 \qquad (10011010010)_2$

$$3^{1234} \equiv 3^{0+2(1+2(0+2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1))))))))))}$$

$$\equiv (3 \cdot 3^{2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1)))))))))})^2$$

$$\equiv (3 \cdot (3^{2(1+2(0+2(1+2(1+2(0+2(0+2(1))))))))})^2)^2$$

$$\equiv (3 \cdot ((3 \cdot 3^{2(0+2(1+2(1+2(0+2(0+2(1)))))))})^2)^2)^2$$

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$$\equiv (3 \cdot ((3 \cdot ((3 \cdot (3 \cdot (3^{2(0+2(1))})^2)^2)^2)^2)^2)^2)^2$$

$$\equiv (3 \cdot ((3 \cdot ((3 \cdot (3 \cdot ((3^{2(1)})^2)^2)^2)^2)^2)^2)^2)^2$$

$$\equiv (3 \cdot ((3 \cdot ((3 \cdot (3 \cdot (((3^{1})^2)^2)^2)^2)^2)^2)^2)^2)^2$$

# Chinese Remainder Theorem (CRT)

- $\forall\ i \neq j \in \{1,2,\ldots k\},\ \gcd(r_i, r_j) = 1,\ 0 \le m_i < r_i$
  Is there an m that satisfies simultaneously the following set of congruence equations?

$$
\begin{aligned}
m &\equiv m_1 \pmod{r_1} \\
&\equiv m_2 \pmod{r_2} \\
&\quad \bullet\ \bullet\ \bullet \\
&\equiv m_k \pmod{r_k}
\end{aligned}
$$

ex: $m \equiv 1 \pmod 3$
$\equiv 2 \pmod 5$
$\equiv 3 \pmod 7$
Note: $\gcd(3,5) = 1$
$\gcd(3,7) = 1$
$\gcd(5,7) = 1$

- 韓信點兵: 三個一數餘一, 五個一數餘二, 七個一數餘三, 請問隊伍中至少有幾名士兵?

# Chinese Remainder Theorem (CRT)

- first solution:

  $n = r_1 r_2 \cdots r_k$

  $z_i = n / r_i$

  $\exists!\ s_i \in Z^*_{r_i}$ s.t. $s_i \cdot z_i \equiv 1 \pmod{r_i}$ (since $\gcd(z_i, r_i) = 1$)

  $m \equiv \sum_{i=1}^{k} z_i \cdot s_i \cdot m_i \pmod{n}$

  Unique solution in $Z_n$?

- ex: $n = 3 \cdot 5 \cdot 7$

  $m_1=1,\ m_2=2,\ m_3=3$

  $r_1=3,\ r_2=5,\ r_3=7$

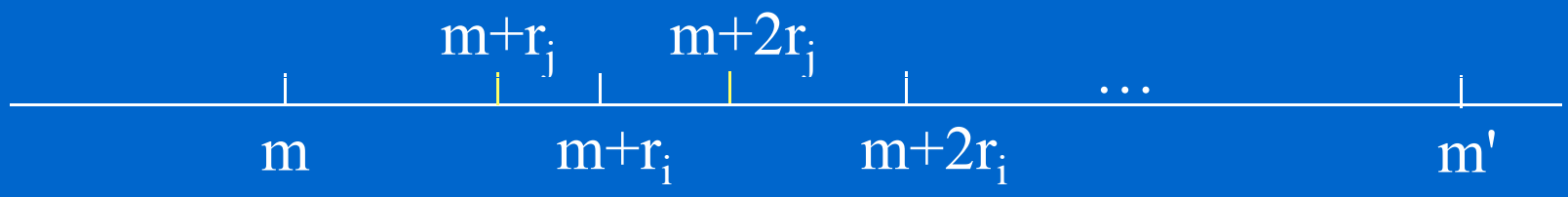
  $z_1=35,\ z_2=21,\ z_3=15$

  $s_1=2,\ s_2=1,\ s_3=1$

  $m \equiv 35 \cdot 2 \cdot 1 + 21 \cdot 1 \cdot 2 + 15 \cdot 1 \cdot 3 \equiv 157 \equiv 52 \pmod{105}$

# Chinese Remainder Theorem (CRT)

- Uniqueness:
  1. If there exists $m' \in Z_n$ ($\neq m$) also satisfies the previous k congruence relations, then
     $$\forall i,\ m' - m \equiv 0 \pmod{r_i}.$$
  2. This is equivalent to $\forall i,\ m' = m + k_i \cdot r_i$




$\Rightarrow$ $m' = m + k \cdot \mathrm{lcm}(r_1, r_2 \ldots r_k) = m + k \cdot n$

$\Rightarrow$ $m' \notin Z_n$ for all $k \neq 0$

contradiction!

# Chinese Remainder Theorem (CRT)

- second solution:

$$R_i = r_1 r_2 \cdots r_{i-1}$$

$$\exists!\ t_i \in Z^*_{r_i} \text{ s.t. } t_i \cdot R_i \equiv 1 \pmod{r_i} \text{ (since } \gcd(R_i, r_i) = 1)$$

$$\begin{cases} \hat{m}_1 = m_1 \\ \hat{m}_i = \hat{m}_{i-1} + R_i \cdot (m_i - \hat{m}_{i-1}) \cdot t_i \pmod{R_{i+1}} & i \geq 2 \\ m = \hat{m}_k \end{cases}$$

**satisfies the first i-1 congruence relations**

Note that

$$\hat{m}_i \equiv m_1 \pmod{r_1}$$
$$\equiv m_2 \pmod{r_2}$$
$$\cdots$$
$$\equiv m_i \pmod{r_i}$$

$m_1=1,\ m_2=2,\ m_3=3$

$r_1=3,\quad r_2=5,\quad r_3=7$

$R_2=3,\ R_3=15,\ R_4=105$

$t_2=2,\quad t_3=1$

ex: $\hat{m}_1 \equiv 1$

$\hat{m}_2 \equiv 1+3\cdot(2-1)\cdot 2=7$

$\hat{m} \equiv m_3 \equiv 7+15\cdot(3-7)\cdot 1$

$\equiv -53 \equiv 52 \pmod{105}$

# Chinese Remainder Theorem (CRT)

- special case:

$$x \equiv m \pmod{r_1} \equiv m \pmod{r_2} \cdots \equiv m_n \pmod{r_n} \Rightarrow x \equiv m \pmod{r_1 r_2 \cdots r_n}$$

- insight of the second solution:

**every step satisfies one more requirement**

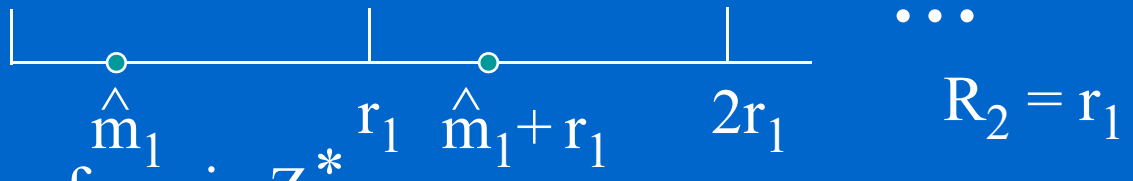
step 1

$x \equiv m_1 \pmod{r_1}$

let $\hat{m}_1 = m_1$

$m_1$ is the only solution for x in $Z^*_{R_2}$

general solution of x must be $\hat{m}_1 + k R_2$ for some k




step 2

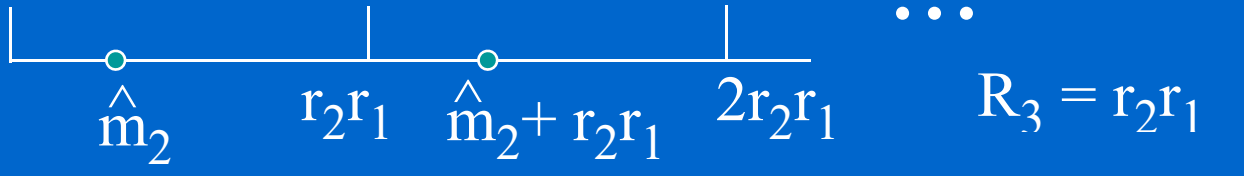
$x \equiv m_1 \pmod{r_1}$

$\equiv m_2 \pmod{r_2}$

let $\hat{m}_2 \equiv \hat{m}_1 + k^* R_2 \pmod{R_3}$ where $k^* = t_2(m_2 - \hat{m}_1)$ and $t_2 R_2 \equiv 1 \pmod{r_2}$

$m_2$ is the only solution for x in $Z^*_{R_3}$

general solution of x must be $\hat{m}_2 + k R_3$ for some k



# Chinese Remainder Theorem (CRT)

- Applications: solve $x^2 \equiv 1 \pmod{35}$
  - $35 = 5 \cdot 7$
  - $x^*$ satisfies $f(x^*) \equiv 0 \pmod{35} \Leftrightarrow$
    $x^*$ satisfies both $f(x^*) \equiv 0 \pmod 5$ and $f(x^*) \equiv 0 \pmod 7$

Proof:

($\Leftarrow$)

$f(x^*) = k_1 \cdot p$ and $f(x^*) = k_2 \cdot q$ imply that
$f(x^*) = k \cdot \text{lcm}(p \cdot q) = k \cdot p \cdot q$ i.e. $f(x^*) \equiv 0 \pmod{p \cdot q}$

($\Rightarrow$)

$f(x^*) = k \cdot p \cdot q$ implies that
$f(x^*) = (k \cdot p) \cdot q = (k \cdot q) \cdot p$ i.e. $f(x^*) \equiv 0 \pmod p$
$\equiv 0 \pmod q$

# Chinese Remainder Theorem (CRT)

- since 5 and 7 are prime, we can solve

  $x^2 \equiv 1 \pmod 5$ and $x^2 \equiv 1 \pmod 7$

  far more easily than $x^2 \equiv 1 \pmod{35}$

  Why?

  - $x^2 \equiv 1 \pmod 5$ has exactly two solutions: $x \equiv \pm 1 \pmod 5$
  - $x^2 \equiv 1 \pmod 7$ has exactly two solutions: $x \equiv \pm 1 \pmod 7$
- put them together and use CRT, there are four solutions
  - $x \equiv 1 \pmod 5 \equiv 1 \pmod 7 \Rightarrow x \equiv 1 \pmod{35}$
  - $x \equiv 1 \pmod 5 \equiv 6 \pmod 7 \Rightarrow x \equiv 6 \pmod{35}$
  - $x \equiv 4 \pmod 5 \equiv 1 \pmod 7 \Rightarrow x \equiv 29 \pmod{35}$
  - $x \equiv 4 \pmod 5 \equiv 6 \pmod 7 \Rightarrow x \equiv 34 \pmod{35}$

# Matlab tools

| | format rat | format long | format long |
|---|---|---|---|
| matrix inverse | inv(A) | | |
| matrix determinant | det(A) | | |
| p = q d + r | r = mod(p, d) or r = rem(p, d) | | |
| | q = floor( p / d ) | | |
| | g = gcd(a, b) | | |
| g = a s + b t | [g, s, t] = gcd(a, b) | | |
| factoring | factor(N) | | |
| prime numbers < N | primes(N) | | |
| test prime | isprime(p) | | |
| mod exponentiation * | powermod(a,b,n) | | |
| find primitive root * | primitiveroot(p) | | |
| crt * | crt([$a_1$ $a_2$ $a_3$...], [$m_1$ $m_2$ $m_3$...]) | | |
| $\phi$(N) * | eulerphi(N) | | |

# Field

- Field: a set that has the operation of addition, multiplication, subtraction, and division by nonzero elements. Also, the associative, commutative, and distributive laws hold.
- Ex. Real numbers, complex numbers, rational numbers, integers mod a prime are fields
- Ex. Integers, 2×2 matrices with real entries are not fields
- Ex. GF(4) = $\{0, 1, \omega, \omega^2\}$
  - $0 + x = x$
  - $x + x = 0$
  - $1 \cdot x = x$
  - $\omega + 1 = \omega^2$
  - Addition and multiplication are commutative and associative, and the distributive law $x(y+z)=xy+xz$ holds for all x, y, z
  - $x^3 = 1$ for all nonzero elements

# Galois Field

- Galois Field: A field with finite element, finite field
- For every power $p^n$ of a prime, there is exactly one finite field with $p^n$ elements (called $GF(p^n)$), and these are the only finite fields.
- For $n > 1$, {integers (mod $p^n$)} do not form a field.
  - Ex. $p \cdot x \equiv 1 \pmod{p^n}$ does not have a solution
    (i.e. p does not have multiplicative inverse)

# How to construct a GF($p^n$)?

- Def: $Z_2[X]$: the set of polynomials whose coefficients are integers mod 2
  - ex. 0, 1, $1+X^3+X^6$…
  - add/subtract/multiply/divide/Euclidean Algorithm: process all coefficients mod 2
    - $(1+X^2+X^4) + (X+X^2) = 1+X+X^4$ bitwise XOR
    - $(1+X+X^3)(1+X) = 1+X^2+X^3+X^4$
    - $X^4+X^3+1 = (X^2+1)(X^2+X+1) + X$ long division

      can be written as

      $X^4+X^3+1 \equiv X \pmod{X^2+X+1}$

# How to construct GF($2^n$)?

- Define $Z_2[X]$ (mod $X^2+X+1$) to be {0, 1, X, X+1}
  - addition, subtraction, multiplication are done mod $X^2+X+1$
  - $f(X) \equiv g(X)$ (mod $X^2+X+1$)
    - if f(X) and g(X) have the same remainder when divided by $X^2+X+1$
    - or equivalently $\exists$ h(X) such that $f(X) - g(X) = (X^2+X+1)\, h(X)$
    - ex. $X \cdot X = X^2 \equiv X+1$ (mod $X^2+X+1$)
  - if we replace X by ω, we can get the same GF(4) as before
  - the modulus polynomial $X^2+X+1$ should be irreducible

Irreducible: polynomial does not factor into polynomials of lower degree with mod 2 arithmetic
ex. $X^2+1$ is not irreducible since $X^2+1 = (X+1)(X+1)$

# How to construct GF($p^n$)?

- $Z_p[X]$ is the set of polynomials with coefficients mod p
- Choose P(X) to be any one irreducible polynomial mod p of degree n (other irreducible P(X)'s would result to isomorphisms)
- Let GF($p^n$) be $Z_p[X]$ mod P(X)

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- An element in $Z_p[X]$ mod P(X) must be of the form

$$a_0 + a_1 X + \ldots + a_{n-1} X^{n-1}$$

each $a_i$ are integers mod p, and have p choices, hence there are $p^n$ possible elements in GF($p^n$)

- multiplicative inverse of any element in GF($p^n$) can be found using extended Euclidean algorithm (over polynomial)

# $GF(2^8)$

- AES (Rijndael) uses $GF(2^8)$ with irreducible polynomial $X^8 + X^4 + X^3 + X + 1$
- each element is represented as
  $b_7 X^7 + b_6 X^6 + b_5 X^5 + b_4 X^4 + b_3 X^3 + b_2 X^2 + b_1 X + b_0$
  each $b_i$ is either 0 or 1
- elements of $GF(2^8)$ can be represented as 8-bit bytes
  $b_7 b_6 b_5 b_4 b_3 b_2 b_1 b_0$
- mod 2 operations can be implemented by XOR in H/W

# $GF(p^n)$

- Definition of generating polynomial g(X) is parallel to the generator in $Z_p$:
  - every element in $GF(p^n)$ (except 0) can be expressed as a power of g(X)
  - the smallest exponent k such that $g(X)^k \equiv 1$ is $p^n - 1$
- Discrete log problem in $GF(p^n)$:
  - given h(X), find an integer k such that
    $$h(X) \equiv g(X)^k \pmod{P(X)}$$
  - believed to be very hard in most situations