Number Theory for Cryptography

密碼學與應用

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Congruence

- **Modulo Operation:**
  - **Question:** What is 12 mod 9?
  - **Answer:** 12 mod 9 ≡ 3 or 12 ≡ 3 (mod 9)
    "12 is congruent to 3 modulo 9"
- **Definition:** Let $a, r, m \in Z$ (where Z is the set of all integers) and $m > 0$. We write
  - $a \equiv r \pmod{m}$ if m divides $a - r$ (i.e. $m \mid a-r$)
  - m is called the *modulus*
  - r is called the *remainder*
  - $a = q \cdot m + r \qquad 0 \le r < m$
- **Example:** $a = 42$ and $m=9$
  - $42 = 4 \cdot 9 + 6$ therefore $42 \equiv 6 \pmod 9$

Greatest Common Divisor

- GCD of a and b is the largest positive integer dividing both a and b
- gcd(a, b) or (a,b)
- ex. gcd(6, 4) = 2, gcd(5, 7) = 1
- Euclidean algorithm   remainder→divisor → dividend → ignore
  - ex. gcd(482, 1180)

$1180 = 2 \cdot 482 + 216$
$482 = 2 \cdot 216 + 50$
$216 = 4 \cdot 50 + 16$
$50 = 3 \cdot 16 + 2$
$16 = 8 \cdot 2 + 0$

gcd

Why does it work?
Let d = gcd(482, 1180)
d | 482 and d | 1180 ⇒ d | 216
because 216 = 1180 - 2 · 482
d | 216 and d | 482 ⇒ d | 50
d | 50 and d | 216 ⇒ d | 16
d | 16 and d | 50 ⇒ d | 2
2 | 16 ⇒ d = 2

Greatest Common Divisor (cont'd)

- Euclidean Algorithm: calculating GCD

gcd(1180, 482)

(輾轉相除法)

2	482 432	1180 964	2
3	50 48	216 200	4
	2	16 16	8
		0	

Greatest Common Divisor (cont'd)

- ✧ Def: a and b are relatively prime: gcd(a, b) = 1
- ✧ Theorem: Let a and b be two integers, with at least one of a, b nonzero, and let d = gcd(a,b). Then there exist integers x, y, gcd(x, y) = 1 such that $a \cdot x + b \cdot y = d$
  - ✶ Constructive proof: Using Extended Euclidean Algorithm to find x and y

$$d = 2 = 50 - 3 \cdot 16$$
$$= (482 - 2 \cdot 216) - 3 \cdot (216 - 4 \cdot 50)$$
$$= \bullet\bullet\bullet\bullet = \underset{a}{1180} \cdot \underset{x}{(-29)} + \underset{b}{482} \cdot \underset{y}{71}$$

$216 = 1180 - 2 \cdot 482$

$50 = 482 - 2 \cdot 216$

$16 = 216 - 4 \cdot 50$

Extended Euclidean Algorithm

Let gcd(a, b) = d

- ✧ Looking for s and t, gcd(s, t) = 1 s.t. $a \cdot s + b \cdot t = d$
- ✧ When d = 1, $t \equiv b^{-1} \pmod a$

$$a = q_1 \cdot b + r_1$$
①
$$b = q_2 \cdot r_1 + r_2$$
② ③
$$r_1 = q_3 \cdot r_2 + r_3$$
④ ⑤
$$r_2 = q_4 \cdot r_3 + \boldsymbol{d}$$

$$r_3 = q_5 \cdot d + 0$$

Ex. $1180 = 2 \cdot 482 + 216$

$1180 - 2 \cdot 482 = 216$

$482 = 2 \cdot 216 + 50$

$482 - 2 \cdot (1180 - 2 \cdot 482) = 50$

$-2 \cdot 1180 + 5 \cdot 482 = 50$

$216 = 4 \cdot 50 + 16$

$(1180 - 2 \cdot 482) -$

$4 \cdot (-2 \cdot 1180 + 5 \cdot 482) = 16$

$9 \cdot 1180 - 22 \cdot 482 = 16$

$50 = 3 \cdot 16 + \boldsymbol{2}$

$(-2 \cdot 1180 + 5 \cdot 482) -$

$3 \cdot (9 \cdot 1180 - 22 \cdot 482) = 2$

$-29 \cdot 1180 + 71 \cdot 482 = 2$

Greatest Common Divisor (cont'd)

- ✶ The above proves only the existence of integers x and y
- ✶ How about gcd(x, y)?

$\in Z$

$$d = a \cdot x + b \cdot y \qquad \Rightarrow \qquad 1 = a/d \cdot x + b/d \cdot y$$
$$d = \gcd(a, b)$$

If gcd(x, y) = r then $1 = a/d \cdot (x' \cdot r) + b/d \cdot (y' \cdot r)$

i.e. $1 = r \cdot (a/d \cdot x' + b/d \cdot y')$

which means that $r \mid 1$ i.e. $r = 1$

gcd(x, y) = 1 ¶

Note: gcd(x, y) = 1 but (x, y) is not unique

e.g. $d = a\,x + b\,y = a\,(x-kb) + b\,(y+ka)$

Greatest Common Divisor (cont'd)

Lemma: gcd(a,b) = gcd(x,y) = gcd(a,y) = gcd(x,b) = 1 $\Leftrightarrow$

$\exists$ a, b, x, y s.t. $1 = a\,x + b\,y$

pf:($\Rightarrow$)

following the previous theorem

($\Leftarrow$)

Given a, b, z, if $\exists$ x, y, gcd(x,y)=1 s.t. z = ax + by then gcd(a, b) | z (also gcd(a, y) | z, gcd(x, b) | z)

(let $d = \gcd(a, b) \Rightarrow d \mid a$ and $d \mid b \Rightarrow d \mid a\,x + b\,y \Rightarrow d \mid z$)

especially, given a, b, $\exists$ x, y s.t. $1 = a\,x + b\,y$

$\Rightarrow \gcd(a, b) \mid 1 \quad \Rightarrow \gcd(a, b) = 1$

Operations under mod n

- Proposition:

  Let a,b,c,d,n be integers with $n \neq 0$, suppose $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$ then

  $a + c \equiv b + d \pmod{n}$,

  $a - c \equiv b - d \pmod{n}$,

  $a \cdot c \equiv b \cdot d \pmod{n}$

- Proposition:

  Let a,b,c,n be integers with $n \neq 0$ and $\gcd(a,n) = 1$.

  If $a \cdot b \equiv a \cdot c \pmod{n}$ then $b \equiv c \pmod{n}$

Operations under mod n

- What is the multiplicative inverse of a (mod n)?

  i.e. $a \cdot a^{-1} \equiv 1 \pmod{n}$ or $a \cdot a^{-1} = 1 + k \cdot n$

  $\gcd(a, n) = 1 \Rightarrow \exists$ s and t such that $a \cdot s + n \cdot t = 1$

  $\Rightarrow a^{-1} \equiv s \pmod{n}$

  This expression also implies gcd(a,n)=1.

- $a \cdot x \equiv b \pmod{n}$, $\gcd(a, n) = 1$, $x \equiv$?

  $x \equiv b \cdot a^{-1} \equiv b \cdot s \pmod{n}$

- $a \cdot x \equiv b \pmod{n}$, $\gcd(a, n) = d > 1$, $x \equiv$? Are there any solutions?

  if $d \mid b$ $(a/d) \cdot x \equiv (b/d) \pmod{n/d}$ $\gcd(a/d,n/d) = 1$

  $x_0 \equiv (b/d) \cdot (a/d)^{-1} \pmod{n/d}$

  $\Rightarrow$ there are d solutions to the equation $a \cdot x \equiv b \pmod{n}$:

  $x_0, x_0+(n/d), \ldots, x_0+(d-1)\cdot(n/d) \pmod{n}$

Matrix inversion under mod n

- A square matrix is invertible mod n if and only if its determinant and n are relatively prime
- ex: in real field R

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

In a finite field Z (mod n)? we need to find the inverse for ad-bc (mod n) in order to calculate the inverse of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \equiv (ad - bc)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \pmod{n}$$

Group

- A group G is a finite or infinite set of elements and a binary operation × which together satisfy

  1. Closure: $\forall$ a,b $\in$G $a \times b = c \in G$ 封閉性
  2. Associativity: $\forall$ a,b,c $\in$G $(a \times b) \times c = a \times (b \times c)$ 結合性
  3. Identity: $\forall$ a $\in$G $1 \times a = a \times 1 = a$ 單位元素
  4. Inverse: $\forall$ a $\in$G $a \times a^{-1} = 1 = a^{-1} \times a$ 反元素

- Abelian group 交換群 $\forall$ a,b $\in$G $a \times b = b \times a$

- Cyclic group G of order m: a group defined by an element $g \in G$ such that $g, g^2, g^3, \ldots, g^m$ are all distinct elements in G (thus cover all elements of G) and $g^m = 1$, the element g is called a generator of G. Ex: Z_n^* (or Z/nZ)

  means $g \times g \times g \times \ldots \times g$

Group (cont'd)

- The **order of a group**: the number of elements in a group G, denoted |G|. If the order of a group is a finite number, the group is said to be a finite group, note $g^{|G|} = 1$ (the identity element).
- The **order of an element g** of a finite group G is the smallest power m such that $g^m = 1$ (the identity element), denoted by $ord_G(g)$
- ex: $\mathbf{Z_n}$: additive group modulo n is the set {0, 1, …, n-1}
  - binary operation: + (mod n)
  - identity: 0
  - inverse: $-x \equiv n-x \pmod n$

  size of Z_n is n, $\underbrace{g+g+\ldots+g}_{} \equiv 0 \pmod n$
- ex: $\mathbf{Z_n^*}$: multiplicative group modulo n is the set {i:0<i<n, gcd(i,n)=1}
  - binary operation: × (mod n)
  - identity: 1
  - inverse: x^{-1} can be found using extended Euclidean Algorithm

  size of Z_n^* is $\phi(n)$, $g^{\phi(n)} \equiv 1 \pmod n$

Ring Z_m

- **Definition:** The ring Z_m consists of
  - The set $Z_m = \{0, 1, 2, \ldots, m-1\}$
  - Two operations "+ (mod m)" and "× (mod m)" for all $a, b \in Z_m$ such that they satisfy the properties on the next slide
- **Example:** $m = 9$ $Z_9 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$
  - $6 + 8 = 14 \equiv 5 \pmod 9$
  - $6 \times 8 = 48 \equiv 3 \pmod 9$

Properties of the ring Z_m

- Consider the ring $Z_m = \{0, 1, \ldots, m-1\}$
  - The additive identity "0": $a + 0 \equiv a \pmod m$
  - The additive inverse of a: $-a = m - a$ s.t. $a + (-a) \equiv 0 \pmod m$
  - Addition is closed i.e if $a, b \in Z_m$ then $a + b \in Z_m$
  - Addition is commutative $a + b \equiv b + a \pmod m$
  - Addition is associative $(a + b) + c \equiv a + (b + c) \pmod m$

  - Multiplicative identity "1": $a \times 1 \equiv a \pmod m$
  - The multiplicative inverse of a exists only when gcd(a,m) = 1 and denoted as a^{-1} s.t. $a^{-1} \times a \equiv 1 \pmod m$ **might or might not exist**
  - Multiplication is closed i.e. if $a, b \in Z_m$ then $a \times b \in Z_m$
  - Multiplication is commutative $a \times b \equiv b \times a \pmod m$
  - Multiplication is associative $(a \times b) \times c \equiv a \times (b \times c) \pmod m$

Some remarks on the ring Z_m

- A ring is an Abelian group under addition and a semigroup under multiplication.
- A semigroup is defined for a set and a binary operator in which the multiplication operation is associative. No other restrictions are placed on a semigroup; thus a semigroup need not have an **identity** element and its elements need not have **inverses** within the semigroup.

Some remarks on the ring Z_m (cont'd)

- Roughly speaking a ring is a mathematical structure in which we can add, subtract, multiply, and even sometimes divide. (A ring in which every element has multiplicative inverse is called a field.)
  - **Example:** Is the division 4/15 (mod 26) possible?

    In fact, $4/15 \bmod 26 \equiv 4 \times 15^{-1} \pmod{26}$

    Does $15^{-1} \pmod{26}$ exist ?

    It exists only if gcd(15, 26) = 1.

    $15^{-1} \equiv 7 \pmod{26}$ therefore,

    $4/15 \bmod 26 \equiv 4 \times 7 \equiv 28 \equiv 2 \bmod 26$

Some remarks on the group Z_m *and* Z_m^*

- The modulo operation can be applied whenever we want

under Z_m

$(\boldsymbol{a} + \boldsymbol{b}) \pmod{\boldsymbol{m}} \equiv [(\boldsymbol{a} \pmod{\boldsymbol{m}}) + ((\boldsymbol{b} \bmod \boldsymbol{m}))] \pmod{\boldsymbol{m}}$

under Z_m^*

$(\boldsymbol{a} \times \boldsymbol{b}) \pmod{\boldsymbol{m}} \equiv [(\boldsymbol{a} \pmod{\boldsymbol{m}}) \times ((\boldsymbol{b} \bmod \boldsymbol{m}))] \pmod{\boldsymbol{m}}$

$\boldsymbol{a}^{\boldsymbol{b}} \pmod{\boldsymbol{m}} \equiv (\boldsymbol{a} \pmod{\boldsymbol{m}})^{\boldsymbol{b}} \pmod{\boldsymbol{m}}$

Question? $\boldsymbol{a}^{\boldsymbol{b}} \pmod{\boldsymbol{m}} \overset{?}{\equiv} \boldsymbol{a}^{(\boldsymbol{b} \bmod \boldsymbol{m})} \pmod{\boldsymbol{m}}$

Exponentiation in Z_m

- Example: $3^8 \pmod 7 \equiv ?$

  $3^8 \pmod 7 \equiv 6561 \pmod 7 \equiv 2$ since $6561 \equiv 937 \times 7 + 2$ or

  $3^8 \pmod 7 \equiv 3^4 \times 3^4 \pmod 7 \equiv 3^2 \times 3^2 \times 3^2 \times 3^2 \pmod 7$

  $\equiv (3^2 \pmod 7)\times(3^2 \pmod 7)\times(3^2 \pmod 7)\times(3^2 \pmod 7)$

  $\equiv 2 \times 2 \times 2 \times 2 \pmod 7 \equiv 16 \pmod 7 \equiv 2$

- The cyclic group Z_m^* and the modulo arithmetic is of central importance to modern public-key cryptography. In practice, the order of the integers involved in PKC are in the range of $[2^{160}, 2^{1024}]$. Perhaps even larger.

Exponentiation in Z_m (cont'd)

- How do we do the exponentiation efficiently?
- $3^{1234} \pmod{789}$ many ways to do this

  a. do 1234 times multiplication and then calculate remainder

  b. repeat 1234 times (multiplication by 3 and calculate remainder)

  c. repeated $\lfloor \log 1234 \rfloor$ times (square, multiply and calculate remainder)

  ex. first tabulate

$3^2 \equiv 9 \pmod{789}$	$3^{32} \equiv 459^2 \equiv 18$	$3^{512} \equiv 732^2 \equiv 93$
$3^4 \equiv 9^2 \equiv 81$	$3^{64} \equiv 18^2 \equiv 324$	$3^{1024} \equiv 93^2 \equiv 759$
$3^8 \equiv 81^2 \equiv 249$	$3^{128} \equiv 324^2 \equiv 39$	
$3^{16} \equiv 249^2 \equiv 459$	$3^{256} \equiv 39^2 \equiv 732$	

$1234 = 1024 + 128 + 64 + 16 + 2$ $(10011010010)_2$

$3^{1234} \equiv 3^{(1024+128+64+16+2)} \equiv (((759 \bullet 39) \bullet 324) \bullet 459) \bullet 9 \equiv 105 \pmod{789}$

Exponentiation in Z_m (cont'd)

calculate $x^y \pmod m$ where $y = b_0 \cdot 2^2 + b_1 \cdot 2 + b_2$

- Method 1:

$$x^{b_2} \Longrightarrow (x^{b_2})\cdot(x^2)^{b_1} \Longrightarrow (x^{b_2}\cdot(x^2)^{b_1})\cdot(x^4)^{b_0}$$

square square

- Method 2:

$$x^{b_0} \Longrightarrow (x^{b_0})^2 \cdot x^{b_1} \Longrightarrow (x^{2\cdot b_0+b_1})^2 \cdot x^{b_2}$$

square square

square and multiply $\lfloor \log y \rfloor$ times

Exponentiation in Z_m (cont'd)

Method 1:

$1234 = 1024 + 128 + 64 + 16 + 2$ $(10011010010)_2$

$$\begin{aligned}
3^{1234} &\equiv 3^{0+2(1+2(0+2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1))))))))))} \\
&\equiv 9 \bullet 9^{2(0+2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1)))))))))} \\
&\equiv 9 \bullet 81^{2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1))))))))} \\
&\equiv 9 \bullet 249^{2(1+2(0+2(1+2(1+2(0+2(0+2(1)))))))} \\
&\equiv 9 \bullet 459 \bullet 459^{2(0+2(1+2(1+2(0+2(0+2(1))))))} \\
&\equiv 9 \bullet 459 \bullet 18^{2(1+2(1+2(0+2(0+2(1)))))} \\
&\equiv 9 \bullet 459 \bullet 324 \bullet 324^{2(1+2(0+2(0+2(1))))} \\
&\equiv 9 \bullet 459 \bullet 324 \bullet 39 \bullet 39^{2(0+2(0+2(1)))} \\
&\equiv 9 \bullet 459 \bullet 324 \bullet 39 \bullet 732^{2(0+2(1))} \\
&\equiv 9 \bullet 459 \bullet 324 \bullet 39 \bullet 93^{2(1)} \\
&\equiv 9 \bullet 459 \bullet 324 \bullet 39 \bullet 759 \bmod 789
\end{aligned}$$

Exponentiation in Z_m (cont'd)

Method 2: $1234 = 1024 + 128 + 64 + 16 + 2$ $(10011010010)_2$

$$\begin{aligned}
3^{1234} &\equiv 3^{0+2(1+2(0+2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1))))))))))} \\
&\equiv (3\bullet 3^{2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1)))))))})^2 \\
&\equiv (3\bullet(3^{2(1+2(0+2(1+2(1+2(0+2(0+2(1)))))))})^2)^2 \\
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&\equiv (3\bullet((3\bullet((3\bullet(3\bullet(3^{2(0+2(1))})^2)^2)^2)^2)^2)^2)^2 \\
&\equiv (3\bullet((3\bullet((3\bullet(3\bullet((3^{2(1)})^2)^2)^2)^2)^2)^2)^2)^2 \\
&\equiv (3\bullet((3\bullet((3\bullet(3\bullet(((3^1)^2)^2)^2)^2)^2)^2)^2)^2)^2
\end{aligned}$$

Chinese Remainder Theorem (CRT)

- $\forall\, i \neq j \in \{1,2,\ldots k\}$, $\gcd(r_i, r_j) = 1$, $0 \le m_i < r_i$
  Is there an m that satisfies simultaneously the following set of congruence equations?

$$\begin{aligned}
m &\equiv m_1 \pmod{r_1} \\
&\equiv m_2 \pmod{r_2} \\
&\bullet\bullet\bullet \\
&\equiv m_k \pmod{r_k}
\end{aligned}$$

ex: $m \equiv 1 \pmod 3$
$\equiv 2 \pmod 5$
$\equiv 3 \pmod 7$
Note: $\gcd(3,5) = 1$
$\gcd(3,7) = 1$
$\gcd(5,7) = 1$

- 韓信點兵: 三個一數餘一, 五個一數餘二, 七個一數餘三, 請問隊伍中至少有幾名士兵?

Chinese Remainder Theorem (CRT)

✧ first solution:

$n = r_1 r_2 \cdots r_k$

$z_i = n / r_i$

$\exists!\ s_i \in Z^*_{r_i}$ s.t. $s_i \cdot z_i \equiv 1 \pmod{r_i}$ (since $\gcd(z_i, r_i) = 1$)

$m \equiv \sum_{i=1}^{k} z_i \cdot s_i \cdot m_i \pmod{n}$

Unique solution in Z_n?

✧ ex: $n = 3 \cdot 5 \cdot 7$

$m_1=1,\ m_2=2,\ m_3=3$

$r_1=3,\ r_2=5,\ r_3=7$

$z_1=35,\ z_2=21,\ z_3=15$

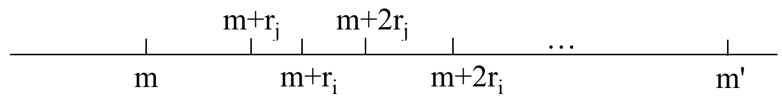
$s_1=2,\ s_2=1,\ s_3=1$

$m \equiv 35\cdot2\cdot1 + 21\cdot1\cdot2 + 15\cdot1\cdot3 \equiv 157 \equiv 52 \pmod{105}$

Chinese Remainder Theorem (CRT)

✧ Uniqueness:

1. If there exists $m' \in Z_n$ ($\neq m$) also satisfies the previous k congruence relations, then
   $\forall i,\ m'-m \equiv 0 \pmod{r_i}$.
2. This is equivalent to $\forall i,\ m' = m + k_i \cdot r_i$



⇨ $m' = m + k \cdot \mathrm{lcm}(r_1, r_2 \ldots r_k) = m + k \cdot n$

⇨ $m' \notin Z_n$ for all $k \neq 0$

contradiction!

Chinese Remainder Theorem (CRT)

✧ second solution:

$R_i = r_1 r_2 \cdots r_{i-1}$

$\exists!\ t_i \in Z^*_{r_i}$ s.t. $t_i \cdot R_i \equiv 1 \pmod{r_i}$ (since $\gcd(R_i, r_i) = 1$)

$$\begin{cases} \hat{m}_1 = m_1 \\ \hat{m}_i = \hat{m}_{i-1} + R_i \cdot (m_i - \hat{m}_{i-1}) \cdot t_i \pmod{R_{i+1}} \quad i \geq 2 \\ m = \hat{m}_k \end{cases}$$

satisfies the first i-1 congruence relations

Note that $\hat{m}_i \equiv m_1 \pmod{r_1}$
$\equiv m_2 \pmod{r_2}$
• • •
$\equiv m_i \pmod{r_i}$

$m_1=1,\ m_2=2,\ m_3=3$
$r_1=3,\ r_2=5,\ r_3=7$
$R_2=3,\ R_3=15,\ R_4=105$
$t_2=2,\ t_3=1$
ex: $\hat{m}_1 \equiv 1$
$\hat{m}_2 \equiv 1+3\cdot(2-1)\cdot2=7$
$\hat{m} \equiv m_3 \equiv 7+15\cdot(3-7)\cdot1$
$\equiv -53 \equiv 52 \pmod{105}$

Chinese Remainder Theorem (CRT)

✧ special case:

$x \equiv m \pmod{r_1} \equiv m \pmod{r_2} \cdots \equiv m_n \pmod{r_n} \Rightarrow x \equiv m \pmod{r_1 r_2 \cdots r_n}$

✧ insight of the second solution: **every step satisfies one more requirement**

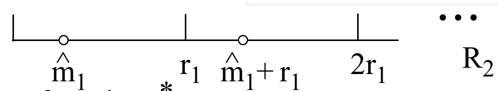
step 1

$x \equiv m_1 \pmod{r_1}$

let $\hat{m}_1 = m_1$

m_1 is the only solution for x in $Z^*_{R_2}$

general solution of x must be $\hat{m}_1 + k R_2$ for some k



$R_2 = r_1$

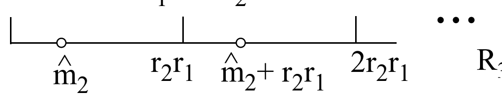
step 2

$x \equiv m_1 \pmod{r_1}$
$\equiv m_2 \pmod{r_2}$

let $\hat{m}_2 \equiv \hat{m}_1 + k^* R_2 \pmod{R_3}$ where $k^* = t_2(m_2 - \hat{m}_1)$ and $t_2 R_2 \equiv 1 \pmod{r_2}$

m_2 is the only solution for x in $Z^*_{R_3}$

general solution of x must be $\hat{m}_2 + k R_3$ for some k



$R_3 = r_2 r_1$

Chinese Remainder Theorem (CRT)

- Applications: solve $x^2 \equiv 1 \pmod{35}$
  - $35 = 5 \cdot 7$
  - x^* satisfies $f(x^*) \equiv 0 \pmod{35}$ $\Leftrightarrow$
    x^* satisfies both $f(x^*) \equiv 0 \pmod 5$ and $f(x^*) \equiv 0 \pmod 7$

    Proof:

    ($\Leftarrow$)

    $f(x^*) = k_1 \cdot p$ and $f(x^*) = k_2 \cdot q$ imply that
    $f(x^*) = k \cdot \mathrm{lcm}(p \cdot q) = k \cdot p \cdot q$ i.e. $f(x^*) \equiv 0 \pmod{p \cdot q}$

    ($\Rightarrow$)

    $f(x^*) = k \cdot p \cdot q$ implies that
    $f(x^*) = (k \cdot p) \cdot q = (k \cdot q) \cdot p$ i.e. $f(x^*) \equiv 0 \pmod p$
    $\equiv 0 \pmod q$

Chinese Remainder Theorem (CRT)

- since 5 and 7 are prime, we can solve
  $x^2 \equiv 1 \pmod 5$ and $x^2 \equiv 1 \pmod 7$
  far more easily than $x^2 \equiv 1 \pmod{35}$ Why?
  - $x^2 \equiv 1 \pmod 5$ has exactly two solutions: $x \equiv \pm 1 \pmod 5$
  - $x^2 \equiv 1 \pmod 7$ has exactly two solutions: $x \equiv \pm 1 \pmod 7$
- put them together and use CRT, there are four solutions
  - $x \equiv 1 \pmod 5 \equiv 1 \pmod 7 \Rightarrow x \equiv 1 \pmod{35}$
  - $x \equiv 1 \pmod 5 \equiv 6 \pmod 7 \Rightarrow x \equiv 6 \pmod{35}$
  - $x \equiv 4 \pmod 5 \equiv 1 \pmod 7 \Rightarrow x \equiv 29 \pmod{35}$
  - $x \equiv 4 \pmod 5 \equiv 6 \pmod 7 \Rightarrow x \equiv 34 \pmod{35}$

Matlab tools

	format rat format long format long
matrix inverse	inv(A)
matrix determinant	det(A)
p = q d + r	r = mod(p, d) or r = rem(p, d)
	q = floor(p / d)
	g = gcd(a, b)
g = a s + b t	[g, s, t] = gcd(a, b)
factoring	factor(N)
prime numbers < N	primes(N)
test prime	isprime(p)
mod exponentiation *	powermod(a,b,n)
find primitive root *	primitiveroot(p)
crt *	crt([a_1 a_2 a_3...], [m_1 m_2 m_3...])
ϕ(N) *	eulerphi(N)

Field

- Field: a set that has the operation of addition, multiplication, subtraction, and division by nonzero elements. Also, the associative, commutative, and distributive laws hold.
- Ex. Real numbers, complex numbers,
  rational numbers, integers mod a prime are fields
- Ex. Integers, 2×2 matrices with real entries are not fields
- Ex. GF(4) = $\{0, 1, \omega, \omega^2\}$
  - $0 + x = x$
  - $x + x = 0$
  - $1 \cdot x = x$
  - $\omega + 1 = \omega^2$

- Addition and multiplication are commutative and associative, and the distributive law x(y+z)=xy+xz holds for all x, y, z
- $x^3 = 1$ for all nonzero elements

Galois Field

- Galois Field: A field with finite element, finite field
- For every power p^n of a prime, there is exactly one finite field with p^n elements (called $GF(p^n)$), and these are the only finite fields.
- For $n > 1$, {integers (mod p^n)} do not form a field.
  - Ex. $p \cdot x \equiv 1 \pmod{p^n}$ does not have a solution (i.e. p does not have multiplicative inverse)

How to construct a $GF(p^n)$?

- Def: $Z_2[X]$: the set of polynomials whose coefficients are integers mod 2
  - ex. 0, 1, $1+X^3+X^6\ldots$
  - add/subtract/multiply/divide/Euclidean Algorithm: process all coefficients mod 2
    - $(1+X^2+X^4) + (X+X^2) = 1+X+X^4$ bitwise XOR
    - $(1+X+X^3)(1+X) = 1+X^2+X^3+X^4$
    - $X^4+X^3+1 = (X^2+1)(X^2+X+1) + X$ long division can be written as $X^4+X^3+1 \equiv X \pmod{X^2+X+1}$

How to construct $GF(2^n)$?

- Define $Z_2[X]$ (mod X^2+X+1) to be {0, 1, X, X+1}
  - addition, subtraction, multiplication are done mod X^2+X+1
  - $f(X) \equiv g(X) \pmod{X^2+X+1}$
    - if f(X) and g(X) have the same remainder when divided by X^2+X+1
    - or equivalently $\exists$ h(X) such that $f(X) - g(X) = (X^2+X+1)\,h(X)$
    - ex. $X \cdot X = X^2 \equiv X+1 \pmod{X^2+X+1}$
  - if we replace X by ω, we can get the same GF(4) as before
  - the modulus polynomial X^2+X+1 should be irreducible

Irreducible: polynomial does not factor into polynomials of lower degree with mod 2 arithmetic
ex. X^2+1 is not irreducible since $X^2+1 = (X+1)(X+1)$

How to construct $GF(p^n)$?

- $Z_p[X]$ is the set of polynomials with coefficients mod p
- Choose P(X) to be any one irreducible polynomial mod p of degree n (other irreducible P(X)'s would result to isomorphisms)
- Let $GF(p^n)$ be $Z_p[X]$ mod P(X)

- An element in $Z_p[X]$ mod P(X) must be of the form
  $$a_0 + a_1 X + \ldots + a_{n-1} X^{n-1}$$
  each a_i are integers mod p, and have p choices, hence there are p^n possible elements in $GF(p^n)$
- multiplicative inverse of any element in $GF(p^n)$ can be found using extended Euclidean algorithm (over polynomial)

GF(2^8)

- AES (Rijndael) uses GF(2^8) with irreducible polynomial $X^8 + X^4 + X^3 + X + 1$
- each element is represented as
  $b_7 X^7 + b_6 X^6 + b_5 X^5 + b_4 X^4 + b_3 X^3 + b_2 X^2 + b_1 X + b_0$
  each b_i is either 0 or 1
- elements of GF(2^8) can be represented as 8-bit bytes
  $b_7b_6b_5b_4b_3b_2b_1b_0$
- mod 2 operations can be implemented by XOR in H/W

GF(p^n)

- Definition of generating polynomial g(X) is parallel to the generator in Z_p:
  - every element in GF(p^n) (except 0) can be expressed as a power of g(X)
  - the smallest exponent k such that $g(X)^k \equiv 1$ is $p^n - 1$
- Discrete log problem in GF(p^n):
  - given h(X), find an integer k such that
    $$h(X) \equiv g(X)^k \pmod{P(X)}$$
  - believed to be very hard in most situations

Recursive GCD

```
int gcd(int p, int q) // assume p >= q
{
    int ans;

    if (p % q == 0)
        ans = q;
    else
        ans = gcd(q, p % q);

    return ans;
}
```

```
int gcd(int p, int q)
{
    int r = p%q;
    if (r == 0)
        return q;
    return gcd(q, r);
}
```


Recursive Extended GCD

- Given $a>b\geq 0$, find g=GCD(a,b) and x, y s.t. $a x + b y = g$ where $|x|\leq b+1$ and $|y|\leq a+1$
- Let $a = q b + r$, $b>r\geq 0 \Rightarrow (q b + r) x + b y = g$
  $\Rightarrow b (q x + y) + r x = g$
  $\Rightarrow b y' + r x = g$, where $y' = q x + y$
- This means that if we can find y' and x satisfying $b y' + (a\%b) x = g$ then x and $y = y' - q x = y' - (a/b) x$ satisfies $a x + b y = g$
  Note that in this way r will eventually be 0

```
void extgcd(int a, int b, int *g, int *x, int *y) { // a > b >=0
    if (b == 0)
        *g = a, *x = 1, *y = 0;
    else {
        extgcd(b, a%b, g, y, x);
        *y = *y - (a/b)*(*x);
    }
}
```