Prime Numbers

密碼學與應用

海洋大學資訊工程系

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  ➢ if b does not have any prime factor less than b, then "b is a prime number that is larger than a_n" … contradiction

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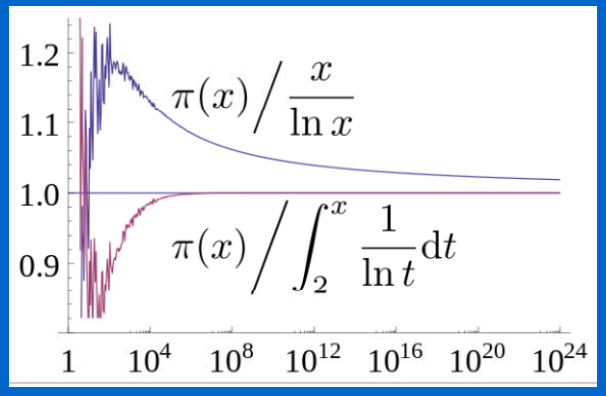

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- Ex: number of 100-digit primes

$$\pi(10^{100}) - \pi(10^{99}) \approx \frac{10^{100}}{\ln 10^{100}} - \frac{10^{99}}{\ln 10^{99}} \approx 3.9 \times 10^{97}$$

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  - Ex. $504 = 2^3 \cdot 3^2 \cdot 7$, $1125 = 3^2 \cdot 5^3$

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  more generally, p is a prime number and $p \mid a \cdot b \cdot \ldots \cdot z$
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    - case 1: $p \mid a$
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      - $p \mid a\,b \Rightarrow p \mid b$

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      - multiply both side by b, $b = \underline{b\,a}\,x + b\,\underline{p}\,y$
      - $p \mid a\,b \Rightarrow p \mid b$
    - In general: if $p \mid a$ then we are done, if $p \nmid a$ then $p \mid bc\ldots z$, continuing this way, we eventually find that p divides one of the factors of the product

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    - assume $n = r_1^{c1} r_2^{c2} \cdots r_k^{ck} p_1^{a1} p_2^{a2} \cdots p_s^{as} = r_1^{c1} r_2^{c2} \cdots r_k^{ck} q_1^{b1} q_2^{b2} \cdots q_t^{bt}$ where p_i, q_j are all distinct primes.

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    - let $m = n / (r_1^{c1} r_2^{c2} \cdots r_k^{ck})$
    - consider p_1 for example, since p_1 divide $m = q_1 q_1 .. q_1 q_2 \cdots q_t$, p_1 must divide one of the factors q_j, contradict the fact that "p_i, q_j are distinct primes"

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("Fair-MAH")

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✡ $1 \cdot 2 \cdot \ldots \cdot (p-1) \equiv \psi(1) \cdot \psi(2) \cdot \ldots \cdot \psi(p-1) \equiv (a \cdot 1) \cdot (a \cdot 2) \cdot \ldots \cdot (a \cdot (p-1))$

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$\equiv a^{p-1} (1 \cdot 2 \cdot \ldots \cdot (p-1)) \pmod{p}$

✡ since gcd(j, p) = 1 for $j \in S$, we can divide both side by 1, 2, 3, … p-1, and obtain $a^{p-1} \equiv 1 \pmod{p}$

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  - ✶ exceptions: $2^{561-1} \equiv 1 \pmod{561}$ although $561 = 3 \cdot 11 \cdot 17$

    $2^{1729-1} \equiv 1 \pmod{1729}$ although $1729 = 7 \cdot 13 \cdot 19$

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    $2^{1729-1} \equiv 1 \pmod{1729}$ although $1729 = 7 \cdot 13 \cdot 19$

  - (*) is a quick test for eliminating composite number

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ex. $\phi(10)=(2-1)\cdot(5-1)=4$ $\phi(120)=120(1-1/2)(1-1/3)(1-1/5)=32$

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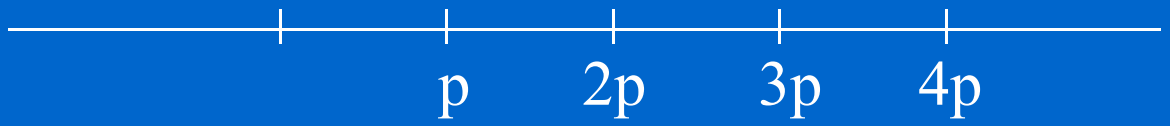

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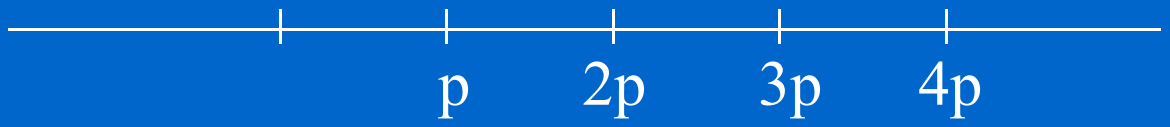

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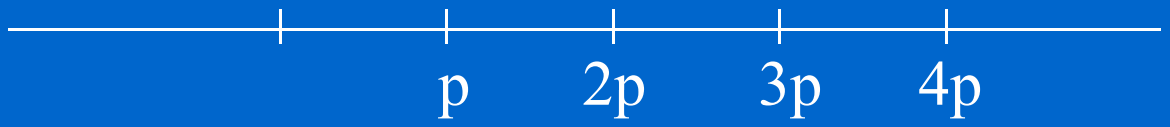

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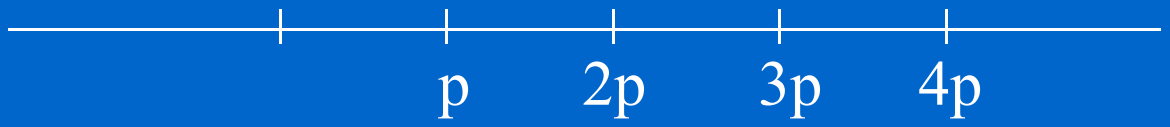

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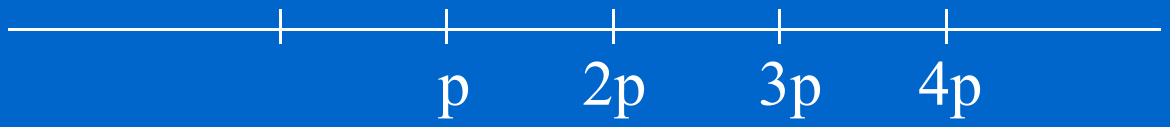

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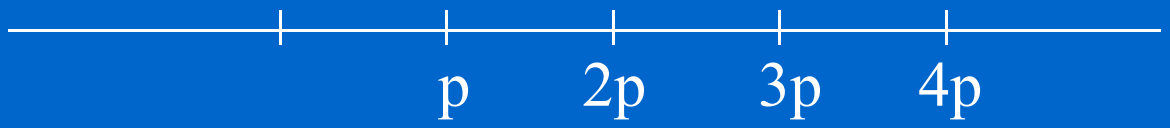

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- Equalities:

$$\frac{1}{1-x} = 1+x+x^2+x^3+\ldots$$

$$1 + 1/2^2 + 1/3^2 + 1/4^2 + 1/5^2 + 1/6^2 + \ldots = \pi^2/6$$

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0.61

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  - Ex. n=4, $\zeta(4) = \pi^4/90 \approx 0.92$

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gcd(a, n)=1 and gcd(x, n)=1
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- $$\prod_{x \in S} x \equiv \prod_{x \in S} \psi(x) \equiv a^{\phi(n)} \prod_{x \in S} x \pmod{n}$$
- since gcd(x, n) = 1 for $x \in S$, we can cancel one by one $x \in S$ of both sides, and obtain $a^{\phi(n)} \equiv 1 \pmod{n}$

Euler's Theorem

true when n is prime

true even when $n = p^k$

- If $\gcd(a,n)=1$ then $a^{\phi(n)} \equiv 1 \pmod{n}$

Proof:

- let S be the set of integers $1 \le x < n$, with $\gcd(x, n) = 1$
- define $\psi(x) \equiv a \cdot x \pmod{n}$ be a mapping $\psi: S \to Z$
- $\forall x \in S$ and $\gcd(a, n) = 1$,

  $\left.\begin{array}{l}\psi(x) \neq 0 \pmod{n} \\ \gcd(\psi(x), n) = 1\end{array}\right\} \Rightarrow \forall x \in S, \psi(x) \in S$, i.e. $\psi: S \to S$
- $\forall\ x, y \in S$, 'if $x \neq y$ then $\psi(x) \not\equiv \psi(y) \pmod{n}$'

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- Example: What are the last three digits of 7^{803}?

  i.e. we want to find $7^{803} \pmod{1000}$

  $1000 = 2^3 \cdot 5^3$, $\phi(1000) = 1000(1-1/2)(1-1/5) = 400$

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- Example: Compute $2^{43210} \pmod{101}$?

  $101 = 1 \cdot 101$, $\phi(101) = 100$

  $2^{43210} \equiv 2^{43210 \pmod{100}} \equiv 2^{10} \equiv 1024 \equiv 14 \pmod{101}$

A second proof of Euler's Theorem

Euler's Theorem: $\forall a \in Z_n^*$, $a^{\phi(n)} \equiv 1 \pmod{n}$

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gcd(p,q)=1

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$$= 1 + p^{r-1} \lambda p + p^{r-1}(p^{r-1}-1)/2 \, (\lambda p)^2 + \ldots$$

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- consider $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, $\phi(n) = n \prod_{\forall p \mid n} (1-1/p)$

Unique Prime Factorization

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  $\Rightarrow a^{n \cdot \lambda(n)} = 1 + n \cdot k \cdot n + \ldots \Rightarrow \forall a \in Z_n^*$ (or $Z_{n^2}^*$), $a^{n \cdot \lambda(n)} \equiv 1 \pmod{n^2}$

Basic Principle to do Exponentiation

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- If you want to work mod n, you should work mod $\phi(n)$ or $\lambda(n)$ in the exponent.

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  - ex. $p=101$, $\phi(p-1)=100\cdot(1-1/2)\cdot(1-1/5)=40$

    $p=143537$, $\phi(p-1)=143536\cdot(1-1/2)\cdot(1-1/8971)=71760$

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$g^{\phi(p)} \equiv g^{ordp(g) * k} g^s \equiv g^s \equiv 1 \pmod p$, but $s < ord_p(g) \Rightarrow s = 0$, i.e. $ord_p(g) \mid \phi(p)$

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Proof:

(a) by definition, $ord_p(g)$ is the smallest positive x s.t. $g^x \equiv 1 \pmod p$

Fermat Theorem: $g^{\phi(p)} \equiv 1 \pmod p$ therefore implies $ord_p(g) \le \phi(p)$

if $\phi(p) = ord_p(g) * k + s$ with $0 \le s < ord_p(g)$

$g^{\phi(p)} \equiv g^{ordp(g) * k} g^s \equiv g^s \equiv 1 \pmod p$, but $s < ord_p(g) \Rightarrow s = 0$, i.e. $ord_p(g) \mid \phi(p)$

(b) assume g is not a primitive root i.e $ord_p(g) < \phi(p)=p-1$

then $\exists$ i, such that $ord_p(g) \mid (p-1)/q_i$ i.e. $g^{(p-1)/qi} \equiv 1 \pmod p$ for some q_i

Primitive Testing Procedure (cont'd)

- Procedure to test a primitive g:

> let p-1 has prime factors $q_1, q_2, \ldots, q_n$, (i.e. $\phi(p)=p-1=q_1^{r_1}...q_n^{r_n}$)
>
> for all q_i, $g^{(p-1)/q_i} \pmod p$ is not $1 \Rightarrow$ g is a primitive

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(c) if for all q_i, $g^{(p-1)/qi} \neq 1 \pmod p$

then $ord_p(g) = \phi(p)$ and g is a primitive root modulo p

Lucas Primality Test

- An integer n is **prime** iff

  $\exists a$, s.t. { **1.** $a^{n-1} \equiv 1 \pmod{n}$
  2. $\forall$ prime factor q of n-1, $a^{n-1/q} \neq 1 \pmod{n}$

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catch: inefficient, factors of n-1 are required

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Proof:

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Fermat's little theorem ensures that "$\forall a \neq kn$, $a^{n-1} \equiv 1 \pmod{n}$"

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the first condition implies that $ord_n(a) \leq n-1$, also, $ord_n(a) \mid n-1$

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the second condition then implies that $ord_n(a) = n-1$ (*)

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the second condition then implies that $ord_n(a) = n-1$ (*)

Euler thm says that $a^{\phi(n)} \equiv 1 \pmod n$, by definition $\phi(n)<n-1$ if n is a composite number, i.e. $ord_n(a) < n-1$, contradict with (*).

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  229 ($a = 6$, $229 - 1 = 2^2 \times 3 \times 19$)

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- example:

  229 ($a = 6$, $229 - 1 = 2^2 \times 3 \times 19$)

verification

$6^{229-1} \equiv 1 \pmod{229}$

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verification

$6^{229-1} \equiv 1 \pmod{229}$

$6^{228/2} \equiv 228 \pmod{229}$

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verification

$6^{229-1} \equiv 1 \pmod{229}$

$6^{228/2} \equiv 228 \pmod{229}$

$6^{228/3} \equiv 134 \pmod{229}$

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$6^{228/2} \equiv 228 \pmod{229}$

$6^{228/3} \equiv 134 \pmod{229}$

$6^{228/19} \equiv 165 \pmod{229}$

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$6^{228/19} \equiv 165 \pmod{229}$

By LPT, if 2, 3, 19 are primes, then 229 is also a prime

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- example:

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  2

verification

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  2 (known prime)

verification

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  2 (known prime)

  3

verification

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  3 ($a = 2, 3 - 1 = 2$)

verification

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  2 (known prime)

  3 ($a = 2, 3 - 1 = 2$)

verification

$2^{3-1} \equiv 1 \pmod 3$

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  2 (known prime)

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verification

$2^{3-1} \equiv 1 \pmod 3$

$2^{2/2} \equiv 2 \pmod 3$

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  2 (known prime)

  3 ($a = 2, 3 - 1 = 2$)

verification

$2^{3-1} \equiv 1 \pmod 3$

$2^{2/2} \equiv 2 \pmod 3$

By LPT, 2 is a prime, then 3 is also a prime

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  19

verification

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  2 (known prime)

  19 ($a = 2, 19 - 1 = 2 \times 3^2$)

verification

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  3 ($a = 2, 3 - 1 = 2$)

  2 (known prime)

  19 ($a = 2, 19 - 1 = 2 \times 3^2$)

verification

$2^{19-1} \equiv 1 \pmod{19}$

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  3 ($a = 2, 3 - 1 = 2$)

  2 (known prime)

  19 ($a = 2, 19 - 1 = 2 \times 3^2$)

verification

$2^{19-1} \equiv 1 \pmod{19}$

$2^{18/2} \equiv 18 \pmod{19}$

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3 ($a = 2, 3 - 1 = 2$)

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19 ($a = 2, 19 - 1 = 2 \times 3^2$)

verification

$2^{19-1} \equiv 1 \pmod{19}$

$2^{18/2} \equiv 18 \pmod{19}$

$2^{18/3} \equiv 7 \pmod{19}$

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verification

$2^{19-1} \equiv 1 \pmod{19}$

$2^{18/2} \equiv 18 \pmod{19}$

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By LPT, if 2 and 3 are primes, then 19 is also a prime

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  3

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verification

$2^{3-1} \equiv 1 \pmod 3$

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Number of Primitive Root in Z_p^*

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  - if gcd(a, p-1)=d, then $(g^a)^{(p-1)/d} \equiv (g^{a/d})^{(p-1)} \equiv 1 \pmod p$ which says that the order of g^a is at most (p-1)/d, therefore, g^a is not a primitive root $\Rightarrow$ There are at most $\phi(p-1)$ primitive roots in Z_p^*

an integer less than p-1

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  - For an element g^a in Z_p^* where gcd(a, p-1) = 1, it is guaranteed that $(g^a)^{(p-1)/q_i} \neq 1 \pmod p$ for all q_i (q_i is factors or p-1)

an integer less than p-1

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  - For an element g^a in Z_p^* where gcd(a, p-1) = 1, it is guaranteed that $(g^a)^{(p-1)/q_i} \neq 1 \pmod p$ for all q_i (q_i is factors or p-1)

    assume that for a certain q_i, $(g^a)^{(p-1)/q_i} \equiv 1 \pmod p$

an integer less than p-1

Number of Primitive Root in Z_p^*

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Note: QR_n is "Quadratic Residue in Z_n^*" to be defined later

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  yes ($\Rightarrow$) $x^2-y=kn=kpq \Rightarrow p \mid x^2-y$ and $q \mid x^2-y$ □

  ($\Leftarrow$) $p \mid x^2-y$ and $q \mid x^2-y$

Finding Square Roots mod n

- Now let's return to the question of solving square roots in Z_n^*, i.e.

  for an integer $y \in QR_n$,

  find $x \in Z_n^*$ such that $x^2 \equiv y \pmod{n}$

- We would like to transform the problem into solving square roots mod p.

- Question: for $n=p \cdot q$

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  ($\Leftarrow$) $p \mid x^2-y$ and $q \mid x^2-y \Rightarrow pq \mid x^2-y$ i.e. $x^2-y=kpq=kn$ □

Finding Square Roots mod p·q

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  - therefore, $p \mid (a-b)$ i.e. $\gcd(a-b, n) = p$ (ex. $\gcd(15-29, 77)=7$)
    $q \mid (a+b)$ i.e. $\gcd(a+b, n) = q$ (ex. $\gcd(15+29, 77)=11$)

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  - $\{g, g^3, \ldots, g^{p-2}\}$ are quadratic non-residues (QNR_p), out of which there are $\phi(p-1)$ primitive roots

Quadratic Residues in Z_p^*

1st proof:

- For each $x \in Z_p^*$, $p-x \neq x \pmod{p}$ (since if x is odd, $p-x$ is even), it's clear that x and $p-x$ are both square roots of a certain $y \in Z_p^*$,

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- Because $|\{g^2, g^4, \ldots, g^{p-1}\}| = (p-1)/2$, there can be no more quadratic residues outside this set. Therefore, the set $\{g, g^3, \ldots, g^{p-2}\}$ contains only quadratic non-residues

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$(g^{2k})^{-1}(g^{2k}) \equiv (g^{2k})^{-1} g \cdot g \cdot \ldots \cdot g \equiv 1 \pmod p$

$\Rightarrow (g^{2k})^{-1} \equiv g^{-1} \cdot g^{-1} \cdot \ldots \cdot g^{-1} \equiv (g^{-1})^{2k} \equiv ((g^{-1})^k)^2$

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- ex. p=143537, p-1=143536=$2^4 \cdot 8971$,

  $\phi(p\text{-}1)=2^4 \cdot 8971 \cdot (1\text{-}1/2) \cdot (1\text{-}1/8971)=71760$ primitives,

  $(p\text{-}1)/2=71768$ QR_p's and 71768 QNR_p's

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  - Elements in Z_p^* can be grouped further according to their order

    since $\forall x \in Z_p^*$, $ord_p(x) \mid p\text{-}1$, we can list all possible orders

$ord_p(x)$	p-1	$\frac{p\text{-}1}{2}$	$\frac{p\text{-}1}{4}$	$\frac{p\text{-}1}{8}$	$\frac{8971}{\frac{p\text{-}1}{16}}$	$\frac{16}{\frac{p\text{-}1}{8971}}$	$\frac{8}{\frac{p\text{-}1}{8971 \cdot 2}}$	$\frac{4}{\frac{p\text{-}1}{8971 \cdot 4}}$	$\frac{2}{\frac{p\text{-}1}{8971 \cdot 8}}$	$\frac{1}{\frac{p\text{-}1}{8971 \cdot 16}}$
	QNR_p	QR_p	QR_p	QR_p	QR_p	QNR_p	QR_p	QR_p	QR_p	QR_p
#	$\phi(p\text{-}1)$					8			2	1

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  ex. $J(7, 143) = J(7, 11) \cdot J(7, 13) = (-1) \cdot (-1) = 1$
  however, there is no integer x such that
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  - Rule 3: J(2, n) = 1 if $(n^2-1)/8$ is even and J(2, n) = -1 otherwise
  - Rule 4: J(a, n) = J(a mod n, n)
  - Rule 5: J(a, b) = J(-a, b) if a <0 and (b-1)/2 is even,
    J(a, b) = -J(-a, b) if a<0 and (b-1)/2 is odd
  - Rule 6: $J(a, b_1 \cdot b_2) = J(a, b_1) \cdot J(a, b_2)$
  - Rule 7: if gcd(a, b)=1, a and b are odd
    - 7a: J(a, b) = J(b, a) if (a-1)·(b-1)/4 is even

Calculation of Jacobi Symbol

- The following algorithm computes the Jacobi symbol J(a, n), for any integer a and odd integer n, recursively:
  - Def 1: J(0, n) = 0 also If n is prime, J(a, n) = 0 if $n|a$
  - Def 2: If n is prime, J(a, n) = 1 if $a \in QR_n$ and J(a, n) = -1 if $a \notin QR_n$
  - Def 3: If n is a composite, $J(a, n) = J(a, p_1 \cdot p_2 \ldots \cdot p_m) = J(a,p_1) \cdot J(a,p_2) \ldots \cdot J(a,p_m)$
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    - 7a: J(a, b) = J(b, a) if (a-1)·(b-1)/4 is even
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QR_n and Jacobi Symbol

- Consider $n = p \cdot q$, where p and q are prime numbers

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Wilson's Theorem

$$(p\text{-}1)! \equiv -1 \pmod{p}$$

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- ✶ Out of the set $\{2, 3, \ldots p\text{-}2\}$, we can form $(p\text{-}3)/2$ pairs such that $i \cdot j \equiv 1 \pmod p$, multiply them together, we obtain $(p\text{-}2)! \equiv 1$

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- From Wilson's theorem, $y^{(p-1)/2} \equiv -1 \pmod{p}$

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Every $y \in QR_p$ has exactly two square roots

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pf: ⋆ $QR_p = \{g^2, g^4, \ldots, g^{p-1}\}$, $|Z_p^*| = p-1$, and $|QR_p| = (p-1)/2$

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- ★ Since $|QR_p| = (p-1)/2$, we can obtain a set of $p-1$ square roots $S = \{g, p-g, g^2, p-g^2, \ldots, g^{(p-1)/2}, p-g^{(p-1)/2}\}$

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- ★ If there is one more square root z of $y \equiv g^{2k}$ which is not g^k and $-g^k$, it must belong to S (which is Z_p^*), say g^j, $j \neq k$, which would imply that $g^{2j} \equiv g^{2k} \pmod p$, and leads to contradiction

Order q Subgroup G_q of Z_p^*

- Let p be a prime number, g be a primitive in Z_p^*

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  a. there are $\phi(p\text{-}1)$ generators in $Z_p^* = \{g^1, g^2, \ldots, g^x, \ldots, g^{p-1}\}$, since $\gcd(p\text{-}1, x) = d > 1$ implies that $\text{ord}_p(g^x) = (p\text{-}1)/d$

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The proof is similar for $G_q \subseteq S$. Therefore, $S = G_q$

Gauss' Lemma

Lemma: let p be a prime, a is an integer s.t. gcd(a, p)=1,

define $\{\alpha_j \equiv j \cdot a \pmod p\}_{j=1,\ldots,(p-1)/2}$,

let n be the number of α_j's s.t. $\alpha_j > p/2$ then $L(a, p) = (-1)^n$

pf.

- ✶ $\alpha_j \in \{r_1, \ldots, r_n\}$ if $\alpha_j > p/2$ and $\alpha_j \in \{s_1, \ldots, s_{(p-1)/2-n}\}$ if $\alpha_j < p/2$
- ✶ Since gcd(a, p)=1, r_i and s_i are all distinct and non-zero
- ✶ Clearly, $0 < p-r_i < p/2$ for i=1,…,n
- ✶ no $p-r_i$ is an s_j: if $p-r_i=s_j$ then $s_j \equiv -r_i \pmod p$

rewrite in terms of a: $u\,a \equiv -v\,a \pmod p$ where $1 \le u, v \le (p-1)/2$

$\Rightarrow u \equiv -v \pmod p$ where $1 \le u, v \le (p-1)/2 \Rightarrow$ impossible

$\Rightarrow \{s_1, \ldots, s_{(p-1)/2-n}, p-r_1, \ldots, p-r_n\}$ is a reordering of $\{1, 2, \ldots, (p-1)/2\}$

- ✶ Thus, $((p-1)/2)! \equiv s_1 \cdots s_{(p-1)/2-n} \cdot (-r_1) \cdots (-r_n) \equiv (-1)^n\, s_1 \cdots s_{(p-1)/2-n} \cdot r_1 \cdots r_n$

$\equiv (-1)^n\, ((p-1)/2)!\, a^{(p-1)/2} \pmod p \Rightarrow L(a, p) = (-1)^n$

Theorem: $J(2, p) = (-1)^{(p^2-1)/8}$

Theorem: let p be a prime, gcd(a, p) = 1 then $L(a, p) = (-1)^t$

where $t = \sum_{j=1}^{(p-1)/2} \lfloor j \cdot a/p \rfloor$. Also $L(2, p) = (-1)^{(p^2-1)/8}$

pf.

- ✶ $\alpha_j \in \{r_1, \ldots, r_n\}$ if $\alpha_j > p/2$ and $\alpha_j \in \{s_1, \ldots, s_{(p-1)/2-n}\}$ if $\alpha_j < p/2$
- ✶ $j\, a = p \lfloor j \cdot a/p \rfloor + \alpha_j$ for $j=1, \ldots, (p-1)/2$

$$\Rightarrow \sum_{j=1}^{(p-1)/2} j\, a = \sum_{j=1}^{(p-1)/2} p \lfloor j \cdot a/p \rfloor + \sum_{j=1}^{n} r_j + \sum_{j=1}^{(p-1)/2-n} s_j$$

- ✶ $\{s_1, \ldots, s_{(p-1)/2-n}, p-r_1, \ldots, p-r_n\}$ is a reordering of $\{1, 2, \ldots, (p-1)/2\}$

$$\Rightarrow \sum_{j=1}^{(p-1)/2} j = \sum_{j=1}^{n} (p-r_j) + \sum_{j=1}^{(p-1)/2-n} s_j = np - \sum_{j=1}^{n} r_j + \sum_{j=1}^{(p-1)/2-n} s_j$$

- ✶ Subtracting the above two equations, we have

$$(a-1) \sum_{j=1}^{(p-1)/2} j = p \left(\sum_{j=1}^{(p-1)/2} \lfloor j \cdot a/p \rfloor - n \right) + 2 \sum_{j=1}^{n} r_j$$

$J(2, p) = (-1)^{(p^2-1)/8}$ (cont'd)

- $\sum_{j=1}^{(p-1)/2} j = 1 + \ldots + (p-1)/2 = (p-1)/2\,(1 + (p-1)/2) / 2 = (p^2-1)/8$
- Thus, we have $(a-1)\,(p^2-1)/8 \equiv \sum_{j=1}^{(p-1)/2} \lfloor j \cdot a/p \rfloor - n \pmod 2$

- If a is odd, $n \equiv \sum_{j=1}^{(p-1)/2} \lfloor j \cdot a/p \rfloor$
- If $a = 2$, $\lfloor j \cdot 2/p \rfloor = 0$ for $j=1, \ldots, (p-1)/2$, $n \equiv (p^2-1)/8 \pmod 2$

therefore, $J(2, p) = (-1)^{(p^2-1)/8}$

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e.g. p = 13

2 is a generator in $Z_{13}^* = \{2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}, 2^{11}, 2^{12}\}$

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Lemma. There are at most $\phi(k)$ ord-k elements in Z_p^*, $k \mid p-1$

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k=12, {2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1}, $\phi(12)$

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Lemma. ord-k elements in $Z_p^* \le \phi(k)$

Lemma. There are at most $\phi(k)$ ord-k elements in Z_p^*, $k \mid p-1$

pf. ✧ Z_p^* is a field $\Rightarrow$ $x^k-1 \equiv 0 \pmod p$ has at most k roots

✧ if $\boldsymbol{a}$ is a nontrivial root ($\boldsymbol{a} \neq 1$), then $\{\boldsymbol{a}^0, \boldsymbol{a}^1, \boldsymbol{a}^2, \ldots, \boldsymbol{a}^{k-1}\}$ is the set of the k distinct roots.

✧ Those a^ℓ with $\gcd(\ell, k) = d > 1$ have order at most **k/d**

✧ Only those a^ℓ with $\gcd(\ell, k) = 1$ might have order k

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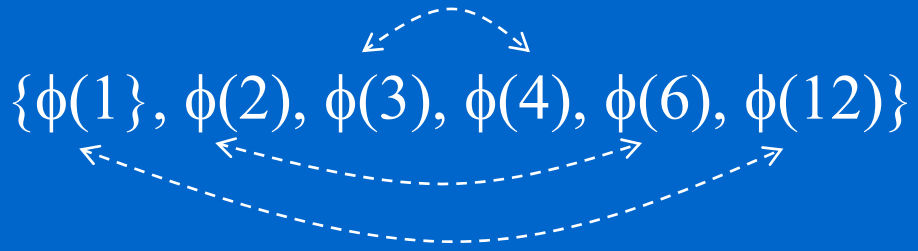

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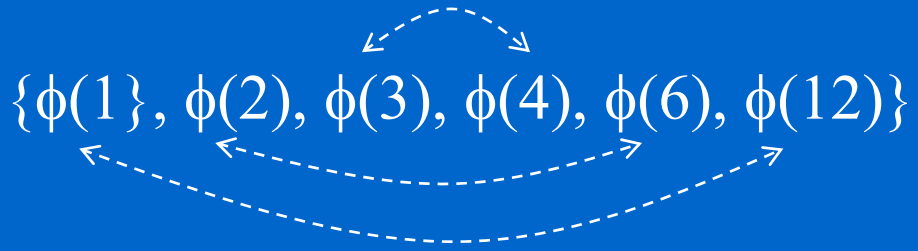

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Ex. p=13, p-1 = |{2,6,11,7}| + |{4,10}| + |{8,5}| + |{3,9}| + |{12}| + |{1}|
k=12 k=6 k=4 k=3 k=2 k=1

Generators in QR_n

- Number of generators in Z_p^*: $\phi(p-1)$

  Let g be a primitive, $Z_p^* = \langle g \rangle = \{g, g^2, g^3, \ldots, g^k, \ldots, g^{p-1}\}$

  if $\gcd(k, p-1) = d \neq 1$ then g^k is not a primitive

  since $(g^k)^{(p-1)/d} = (g^{k/d})^{p-1} = 1$, i.e. $ord_p(g^k) \leq (p-1)/d$

  if $\gcd(k, p-1) = 1$ and g^k is not a primitive, then $d = ord_p(g^k) < p-1$, i.e.

  $(g^k)^d = 1$; g is a primitive $\Rightarrow p-1 \mid k\,d \Rightarrow p-1 \mid d$ contradiction.

- Z_n^* is not a cyclic group ($n = p\,q$, $p=2p'+1$, $q=2q'+1$, $\lambda(n)=2p'q'$)

  Since $x^{\lambda(n)} \equiv 1 \pmod n$, there is no generator that can generate all members in Z_n^*

- QR_n is a cyclic group of order $\lambda(n)/2 = \text{lcm}(p-1, q-1)/2 = p'\,q'$

  $\forall\, x \in Z_n^*$, $x^{\lambda(n)} \equiv 1 \pmod n$ Carmichael's Theorem

  clearly, $(x^2)^{\lambda(n)/2} \equiv 1 \pmod n$, $QR_n = \{x^2 \mid \forall\, x \in Z_n^*\}$

  i.e. $\forall\, y \in QR_n$, $ord_n(y) \mid p'\,q'$ $(ord_n(y) \in \{1, p', q', p'q'\})$

Generators in QR_n (cont'd)

cyclic? $\exists\, x^* \in Z_n^*\ ord_n(x^*) = \lambda(n) = 2\,p'\,q' \Rightarrow$

$\exists\, y^*\ (=(x^*)^2) \in QR_n$ s.t. $ord_n(y^*) = \lambda(n)/2 = p'\,q'$

- Let y be a random element in QR_n, the probability that y is a generator is close to 1

Let y^* be a generator of QR_n,

$$QR_n = \langle y^* \rangle = \{y^*, (y^*)^2, (y^*)^3, \ldots, (y^*)^k, \ldots, (y^*)^{p'q'}\}$$

if $\gcd(k, p'q') = d \neq 1$ then $(y^*)^k$ is not a generator

since $((y^*)^k)^{p'q'/d} = ((y^*)^{k/d})^{p'q'} = 1$, i.e. $ord_p((y^*)^k) \leq (p'q')/d$

$\phi(p'q') = \phi(p')\,\phi(q') = (p'-1)(q'-1) = p'q' - p' - q' + 1$

$= p'q' - (p'-1) - (q'-1) - 1$

$\forall\, x \in \{(y^*)^{q'}, (y^*)^{2q'}, \ldots, (y^*)^{(p'-1)q'}\}\ ord_n(x) = p'$

$\forall\, x \in \{(y^*)^{p'}, (y^*)^{2p'}, \ldots, (y^*)^{(q'-1)p'}\}\ ord_n(x) = q'$

$ord_n(1) = 1$

$Pr\{x \text{ is a generator} \mid x \in_R QR_n\} = \phi(p'q') / (p'q')$ is close to 1

Subgroups in Z_n^*

Consider $n = p\,q$, $p=2p'+1$, $q=2q'+1$, $m=p'q'$, $\lambda(n) = \text{lcm}(p-1, q-1)=2m$,

$\phi(n) = (p-1)(q-1) = 4m$

- **Z_n^*** is not a cyclic group
  - Carmichael's theorem asserts that no element in Z_n^* can generate all elements in Z_n^*. (maximum order is 2m instead of 4m)
  - However, Z_n^* is still a group over modulo n multiplication.
- **QR_n** is a cyclic subgroup of order $m = \lambda(n)/2$, $QR_n = \{x^2 \mid \forall\ x \in Z_n^*\}$
  - $J_{00} = \{x \in Z_n^* \mid J(x,p)=1 \text{ and } J(x,q)=1\}$
  - If there exists an element in Z_n^* whose order is 2m, then QR_n is clearly a cyclic group. (Will the precondition be true?)
  - $\forall\ x \in Z_n^*\ x^{2m} \equiv 1 \pmod n$ implies that $\forall\ y \in QR_n\ \text{ord}_n(y) \mid p'q'$ i.e. $\text{ord}_n(y)$ is either 1, p', q', or p'q' (if there is one y s.t. $\text{ord}_n(y)=m$ then y is a generator and QR_n is cyclic). Let's construct one.

Subgroups in Z_n^* (cont'd)

Let g_1 be a generator in Z_p^*, and g_2 be a generator in Z_q^*

Let **$g \equiv g_1 \pmod p \equiv g_2 \pmod q$**, (note that $J(g, n) = 1$, $g \in J_{11}$)

$g^{p-1} \equiv g^{2p'} \equiv {g_1}^{2p'} \equiv 1 \pmod p$, $g^{q-1} \equiv g^{2q'} \equiv {g_2}^{2q'} \equiv 1 \pmod q$

$\Rightarrow$ $g^{2p'q'} \equiv 1 \pmod p$ and $g^{2q'p'} \equiv 1 \pmod q$ i.e. $g^{2p'q'} \equiv 1 \pmod n$

if there exists a $k \in \{1, 2, p', q', 2p', 2q', p'q'\}$ s.t. $g^k \equiv 1 \pmod n$

then $\text{ord}_n(g)$ is not $2p'q'$

1. $k=1$: $\Rightarrow g_1 \equiv 1 \pmod p$ contradict with $\text{ord}_p(g_1) = p-1$
2. $k=p'$: $\Rightarrow g^{p'} \equiv {g_1}^{p'} \equiv 1 \pmod p$ contradict with $\text{ord}_p(g_1) = 2p'$
3. $k=q'$: $\Rightarrow g^{q'} \equiv {g_2}^{q'} \equiv 1 \pmod q$ contradict with $\text{ord}_q(g_2) = 2q'$
4. $k=2$: $\Rightarrow {g_1}^2 \equiv 1 \pmod p$ contradict with $\text{ord}_p(g_1) = p-1$
5. $k=2p'$: $\Rightarrow g^{2p'} \equiv {g_2}^{2p'} \equiv 1 \pmod q$ contradict with $\text{ord}_q(g_2) = 2q'$
6. $k=2q'$: $\Rightarrow g^{2q'} \equiv {g_1}^{2q'} \equiv 1 \pmod p$ contradict with $\text{ord}_p(g_1) = 2p'$

Subgroups in Z_n^* (cont'd)

7. $k=p'q'$: $\Rightarrow g^{p'q'} \equiv g_1^{p'q'} \equiv 1 \pmod p$

since $g_1^{2p'} \equiv 1 \pmod p$ and

$\gcd(q', 2) = 1 \Rightarrow \exists\ a, b$ s.t. $a\,q' + b\,2 = 1$

$\Rightarrow g_1^{p'} \equiv g_1^{p'(a\,q' + b\,2)} \equiv (g_1^{p'\,q'})^a (g_1^{2\,p'})^b \equiv 1 \pmod p$

contradict with $\mathrm{ord}_p(g_1) = 2p'$

1~7 implies that $\mathrm{ord}_n(g) = 2p'q'$, i.e. $QR_o = \{g^2, g^4, \ldots, g^{p'q'}\}$ and QR_n is a cyclic group.

- ★ Pr{Elements in QR_n being a generator} = $\phi(p'q') / (p'q')$

✧ **J_n** is a cyclic subgroup of order $2m = \lambda(n)$, $J_n = \{x \in Z_n^* \mid J(x,n)=1\}$

- ★ $J_{11} = \{x \in Z_n^* \mid J(x,p)=-1 \text{ and } J(x,q)=-1\}$
- ★ The above proof also shows that $J_n = \{g, g^2, \ldots, g^{2p'q'}\}$ is cyclic
- ★ Pr{Elements in J_n being a generator} = $\phi(p'q') / (2p'q')$

✧ **$J_{01} \cup J_{10}$** $= Z_n^* \setminus \{J_{00} \cup J_{11}\}$ is not a subgroup in Z_n^*

- ★ if $x \in J_{01}$ then $x * x \in J_{00}$

Generator in QR_n

- $n = p\,q$, $p=2p'+1$, $q=2q'+1$
- Find a generator in QR_n

1. Find a generator g_1 of Z_p^* (i.e. $Z_p^* = \langle g_1 \rangle$) and g_2 of Z_q^* (i.e. $Z_q^* = \langle g_2 \rangle$)
2. Calculate the generator $h_1 \equiv g_1^2 \pmod p$ of QR_p and $h_2 \equiv g_2^2 \pmod 1$ of QR_q
3. Let $h \equiv h_1 \pmod p \equiv h_2 \pmod q$.

It is clear that $h \equiv g^2 \pmod n$, i.e. $h \in QR_n$, where $g \equiv g_1 \pmod p \equiv g_2 \pmod q$.

Claim: h is a generator of QR_n

pf.

$y \in QR_n \Rightarrow y \in QR_p$ and $y \in QR_q$

i.e. $\exists\, x_1 \in Z_{p'}$ and $x_2 \in Z_{q'}$, $y \equiv h_1^{x_1} \pmod p \equiv h_2^{x_2} \pmod q$

$\Rightarrow y \equiv g_1^{2x_1} \pmod p \equiv g_2^{2x_2} \pmod q$

$\Rightarrow y \equiv g^{2x} \pmod n$ if $2x \equiv 2x_1 \pmod{p-1} \equiv 2x_2 \pmod{q-1}$

a unique $x \in Z_{p'q'}$ exists by CRT since $\gcd(p-1, q-1) = \gcd(2p', 2q') = 2$

$\Rightarrow y \equiv h^x \pmod n$

Generate Elements in Z_n^*

- Z_n^* is NOT a cyclic group ($n = p\,q$, $p=2p'+1$, $q=2q'+1$, $m=p'\,q'$)
- How do we generate random elements in Z_n^*?

  $Z_n^* = \{ g^a u^{-e\,b_1} (-1)^{b_2} \mid g$ is a generator in QR_n, $\gcd(e, \phi(n)) = 1$,
  $u \in_R Z_n^*$ and $J(u,n) = -1$,
  $a \in \{0,\ldots,m-1\}$, $b_1 \in \{0,1\}$, and $b_2 \in \{0,1\}$ $\}$

  Note: 1. $J(-1, n) = 1$ and $-1 \in J_n \backslash QR_n$ since $(-1)^{(p-1)/2} \equiv (-1)^{p'} \equiv -1 \pmod p$
  2. e is odd, $\phi(n)-e$ is also odd, $J(u^{-e}, n) = J(u, n) = -1$

- We can view the above as 4 parts

  1. J_{00} (QR_n): $b_1 = b_2 = 0$, $J_{00} = \{g^a \mid a \in \{0,\ldots,m-1\}\}$
  2. J_{11} ($J_n \backslash QR_n$): $b_1 = 0$, $b_2 = 1$, $J_{11} = \{-g^a \mid a \in \{0,\ldots,m-1\}\}$

  Assume that $J(u, p) = -1$ and $J(u, q) = 1$

  3. J_{01}: $b_1 = 1$, $b_2 = 0$, $J_{01} = \{g^a u^{-e} \mid a \in \{0,\ldots,m-1\}\}$
  4. J_{10}: $b_1 = 1$, $b_2 = 1$, $J_{01} = \{-g^a u^{-e} \mid a \in \{0,\ldots,m-1\}\}$

Lagrange's Theorem

- Theorem: for any finite group G, the order (number of elements) of every subgroup H of G divides the order of G.
  - proof sketch: divide G into left cosets H – equivalence classes, and show that they have the same size.

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- Any prime order group is cyclic.