Number Theory for Cryptography

密碼學與應用

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Congruence

- **Modulo Operation:**
  - **Question:** What is 12 mod 9?
  - **Answer:** 12 mod 9 $\equiv$ 3 or 12 $\equiv$ 3 (mod 9)

    "12 is congruent to 3 modulo 9"

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- **Definition:** Let a, r, $m \in Z$ (where Z is the set of all integers) and $m > 0$. We write
  - $a \equiv r \pmod{m}$ if m divides $a - r$ (i.e. m | a-r)
  - m is called the *modulus*
  - r is called the *remainder*
  - $a = q \cdot m + r \qquad 0 \leq r < m$

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  - $a = q \cdot m + r \qquad 0 \leq r < m$
- **Example:** $a = 42$ and $m=9$
  - $42 = 4 \cdot 9 + 6$ therefore $42 \equiv 6 \pmod{9}$

Greatest Common Divisor

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$$482 = 2 \cdot 216 + 50$$

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$$216 = 4 \cdot 50 + 16$$

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$$1180 = 2 \cdot 482 + 216$$
$$482 = 2 \cdot 216 + 50$$
$$216 = 4 \cdot 50 + 16$$
$$50 = 3 \cdot 16 + \mathbf{2}$$

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$$1180 = 2 \cdot 482 + 216$$
$$482 = 2 \cdot 216 + 50$$
$$216 = 4 \cdot 50 + 16$$
$$50 = 3 \cdot 16 + 2$$
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$$1180 = 2 \cdot 482 + 216$$
$$482 = 2 \cdot 216 + 50$$
$$216 = 4 \cdot 50 + 16$$
$$50 = 3 \cdot 16 + 2$$
$$16 = 8 \cdot 2 + 0$$

gcd

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$1180 = 2 \cdot 482 + 216$

$482 = 2 \cdot 216 + 50$

$216 = 4 \cdot 50 + 16$

$50 = 3 \cdot 16 + 2$ gcd

$16 = 8 \cdot 2 + 0$

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$$482 = 2 \cdot 216 + 50$$
$$216 = 4 \cdot 50 + 16$$
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$$16 = 8 \cdot 2 + 0$$

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$$1180 = 2 \cdot 482 + 216$$
$$482 = 2 \cdot 216 + 50$$
$$216 = 4 \cdot 50 + 16$$
$$50 = 3 \cdot 16 + 2$$
$$16 = 8 \cdot 2 + 0$$

gcd

Why does it work?

Let d = gcd(482, 1180)

$d \mid 482$ and $d \mid 1180 \Rightarrow d \mid 216$

because $216 = 1180 - 2 \cdot 482$

$d \mid 216$ and $d \mid 482 \Rightarrow d \mid 50$

$d \mid 50$ and $d \mid 216 \Rightarrow d \mid 16$

$d \mid 16$ and $d \mid 50 \Rightarrow d \mid 2$

$2 \mid 16 \Rightarrow d = 2$

Greatest Common Divisor (cont'd)

- Euclidean Algorithm: calculating GCD

  gcd(1180, 482)

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  gcd(1180, 482)

1180

Greatest Common Divisor (cont'd)

- Euclidean Algorithm: calculating GCD

  gcd(1180, 482)

482	1180

Greatest Common Divisor (cont’d)

- Euclidean Algorithm: calculating GCD

  gcd(1180, 482)

482	1180	2

Greatest Common Divisor (cont'd)

✧ Euclidean Algorithm: calculating GCD

gcd(1180, 482)

482	1180	2
	964	

Greatest Common Divisor (cont'd)

- Euclidean Algorithm: calculating GCD

gcd(1180, 482)

482	1180	2
	964	
	216	

Greatest Common Divisor (cont'd)

- Euclidean Algorithm: calculating GCD

gcd(1180, 482)

2	482	1180	2
		964	
		216	

Greatest Common Divisor (cont'd)

✧ Euclidean Algorithm: calculating GCD

gcd(1180, 482)

2	482	1180	2
	432	964	
		216	

Greatest Common Divisor (cont'd)

✧ Euclidean Algorithm: calculating GCD

gcd(1180, 482)

2	482	1180	2
	432	964	
	50	216	

Greatest Common Divisor (cont'd)

- Euclidean Algorithm: calculating GCD

gcd(1180, 482)

2	482	1180	2
	432	964	
	50	216	4

Greatest Common Divisor (cont'd)

- Euclidean Algorithm: calculating GCD

gcd(1180, 482)

2	482	1180	2
	432	964	
	50	216	4
		200	

Greatest Common Divisor (cont'd)

- Euclidean Algorithm: calculating GCD

gcd(1180, 482)

2	482	1180	2
	432	964	
	50	216	4
		200	
		16	

Greatest Common Divisor (cont'd)

✧ Euclidean Algorithm: calculating GCD

gcd(1180, 482)

2	482 432	1180 964	2
3	50	216 200	4
		16	

Greatest Common Divisor (cont’d)

✧ Euclidean Algorithm: calculating GCD

gcd(1180, 482)

2	482 432	1180 964	2
3	50 48	216 200	4
		16	

Greatest Common Divisor (cont'd)

- Euclidean Algorithm: calculating GCD

gcd(1180, 482)

2	482 432	1180 964	2
3	50 48	216 200	4
	2	16	

Greatest Common Divisor (cont'd)

✧ Euclidean Algorithm: calculating GCD

gcd(1180, 482)

2	482 432	1180 964	2
3	50 48	216 200	4
	2	16	8

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gcd(1180, 482)

2	482	1180	2
	432	964	
3	50	216	4
	48	200	
	2	16	8
		16	

Greatest Common Divisor (cont'd)

- Euclidean Algorithm: calculating GCD

  gcd(1180, 482)

2	482	1180	2
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3	50	216	4
	48	200	
	2	16	8
		16	
		0	

Greatest Common Divisor (cont’d)

- Euclidean Algorithm: calculating GCD

gcd(1180, 482)

2	482 432	1180 964	2
3	50 48	216 200	4
	2	16 16	8
		0	

Greatest Common Divisor (cont’d)

✧ Euclidean Algorithm: calculating GCD

gcd(1180, 482)

(輾轉相除法)

2	482	1180	2
	432	964	
3	50	216	4
	48	200	
	2	16	8
		16	
		0	

Greatest Common Divisor (cont'd)

- Def: a and b are relatively prime: $\gcd(a, b) = 1$
- Theorem: Let a and b be two integers, with at least one of a, b nonzero, and let $d = \gcd(a,b)$. Then there exist integers x, y, $\gcd(x, y) = 1$ such that $a \cdot x + b \cdot y = d$
  - Constructive proof: Using Extended Euclidean Algorithm to find x and y

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$d = 2 = 50 - 3 \cdot 16$

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$d = 2 = 50 - 3 \cdot 16$

$50 = 482 - 2 \cdot 216$

$16 = 216 - 4 \cdot 50$

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$d = 2 = 50 - 3 \cdot 16$

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$$d = 2 = 50 - 3 \cdot 16$$

$$= (482 - 2 \cdot 216) - 3 \cdot (216 - 4 \cdot 50)$$

$$50 = 482 - 2 \cdot 216$$

$$16 = 216 - 4 \cdot 50$$

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$$d = 2 = 50 - 3 \cdot 16$$
$$= (482 - 2 \cdot 216) - 3 \cdot (216 - 4 \cdot 50)$$

$$216 = 1180 - 2 \cdot 482$$
$$50 = 482 - 2 \cdot 216$$
$$16 = 216 - 4 \cdot 50$$

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$d = 2 = 50 - 3 \cdot 16$

$= (482 - 2 \cdot 216) - 3 \cdot (216 - 4 \cdot 50)$

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$$d = 2 = 50 - 3 \cdot 16$$

$$= (482 - 2 \cdot 216) - 3 \cdot (216 - 4 \cdot 50)$$

$$= \bullet \bullet \bullet \bullet = 1180 \cdot (-29) + 482 \cdot 71$$

$$216 = 1180 - 2 \cdot 482$$

$$50 = 482 - 2 \cdot 216$$

$$16 = 216 - 4 \cdot 50$$

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  - Constructive proof: Using Extended Euclidean Algorithm to find x and y

$$216 = 1180 - 2 \cdot 482$$
$$50 = 482 - 2 \cdot 216$$
$$16 = 216 - 4 \cdot 50$$

$$\begin{aligned} d = 2 &= 50 - 3 \cdot 16 \\ &= (482 - 2 \cdot 216) - 3 \cdot (216 - 4 \cdot 50) \\ &= \bullet \bullet \bullet \bullet = \underset{a}{1180} \cdot \underset{x}{(-29)} + \underset{b}{482} \cdot \underset{y}{71} \end{aligned}$$

Extended Euclidean Algorithm

Let gcd(a, b) = d

- ✧ Looking for s and t, gcd(s, t) = 1 s.t. $a \cdot s + b \cdot t = d$
- ✧ When d = 1, $t \equiv b^{-1} \pmod{a}$

$$a = q_1 \cdot b + r_1$$

① ②

$$b = q_2 \cdot r_1 + r_2$$

③ ④

$$r_1 = q_3 \cdot r_2 + r_3$$

⑤

$$r_2 = q_4 \cdot r_3 + \boldsymbol{d}$$

$$r_3 = q_5 \cdot d + 0$$

Ex.

$$1180 = 2 \cdot 482 + 216$$
$$1180 - 2 \cdot 482 = 216$$
$$482 = 2 \cdot 216 + 50$$
$$482 - 2 \cdot (1180 - 2 \cdot 482) = 50$$
$$-2 \cdot 1180 + 5 \cdot 482 = 50$$
$$216 = 4 \cdot 50 + 16$$
$$(1180 - 2 \cdot 482) - 4 \cdot (-2 \cdot 1180 + 5 \cdot 482) = 16$$
$$9 \cdot 1180 - 22 \cdot 482 = 16$$
$$50 = 3 \cdot 16 + \boldsymbol{2}$$
$$(-2 \cdot 1180 + 5 \cdot 482) - 3 \cdot (9 \cdot 1180 - 22 \cdot 482) = 2$$
$$-29 \cdot 1180 + 71 \cdot 482 = 2$$

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$$d = a \cdot x + b \cdot y$$

$$d = \gcd(a, b)$$

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- How about gcd(x, y)?

$$\left.\begin{array}{l} d = a \cdot x + b \cdot y \\ d = \gcd(a, b) \end{array}\right. \quad \Rightarrow \quad 1 = a/d \cdot x + b/d \cdot y$$

Greatest Common Divisor (cont'd)

- The above proves only the existence of integers x and y
- How about gcd(x, y)?

$$d = a \cdot x + b \cdot y \qquad \Rightarrow \qquad 1 = \underbrace{a/d}_{\in Z} \cdot x + \underbrace{b/d}_{\in Z} \cdot y$$
$$d = \gcd(a, b)$$

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If $\gcd(x, y) = r$, $r \geq 1$ then

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If $\gcd(x, y) = r$, $r \geq 1$ then

$$r \mid x \text{ and } r \mid y$$

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If $\gcd(x, y) = r$, $r \geq 1$ then

$$r \mid x \text{ and } r \mid y \;\Rightarrow\; r \mid a/d \cdot x + b/d \cdot y$$

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$$d = a \cdot x + b \cdot y \qquad \Rightarrow \qquad 1 = a/d \cdot x + b/d \cdot y$$
$$d = \gcd(a, b)$$

$\in Z$

If $\gcd(x, y) = r$, $r \geq 1$ then

$$r \mid x \text{ and } r \mid y \;\Rightarrow\; r \mid a/d \cdot x + b/d \cdot y$$

which means that $r \mid 1$ i.e. $r = 1$

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$$\gcd(x, y) = 1 \qquad \P$$

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$\in Z$

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If $\gcd(x, y) = r$, $r \geq 1$ then

$$r \mid x \text{ and } r \mid y \;\Rightarrow\; r \mid a/d \cdot x + b/d \cdot y$$

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$\gcd(x, y) = 1$ ¶

Note: gcd(x, y) = 1 but (x, y) is not unique

e.g. $d = a\,x + b\,y = a\,(x-k\cdot b) + b\,(y+k\cdot a)$

when k increases, $x-k\cdot b$ decreases and become negative

Greatest Common Divisor (cont'd)

Lemma: $\gcd(a,b) = \gcd(x,y) = \gcd(a,y) = \gcd(x,b) = 1 \Leftrightarrow$

$\exists$ a, b, x, y s.t. $1 = a\,x + b\,y$

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pf:

($\Rightarrow$) following the previous theorem

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($\Leftarrow$) let $d = \gcd(a, b)$, $d \geq 1$

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pf:

($\Rightarrow$) following the previous theorem

($\Leftarrow$) let d = gcd(a, b), d $\geq$ 1

$\Rightarrow$ d | a and d | b

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($\Leftarrow$) let d = gcd(a, b), $d \geq 1$

$\Rightarrow$ d | a and d | b

$\Rightarrow$ $d \mid a\,x + b\,y$

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($\Leftarrow$) let $d = \gcd(a, b)$, $d \geq 1$

$\Rightarrow d \mid a$ and $d \mid b$

$\Rightarrow d \mid a\,x + b\,y \quad = 1$

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($\Leftarrow$) let $d = \gcd(a, b)$, $d \geq 1$

$\Rightarrow d \mid a$ and $d \mid b$

$\Rightarrow d \mid a\,x + b\,y \quad = 1$

$\Rightarrow d = 1$

Greatest Common Divisor (cont'd)

Lemma: gcd(a,b) = gcd(x,y) = gcd(a,y) = gcd(x,b) = 1 $\Leftrightarrow$

$\exists$ a, b, x, y s.t. $1 = a\,x + b\,y$

pf:

($\Rightarrow$) following the previous theorem

($\Leftarrow$) let d = gcd(a, b), d $\geq$ 1

$\Rightarrow$ d | a and d | b

$\Rightarrow$ d | a x + b y $\quad = 1$

$\Rightarrow$ d = 1

similarly, gcd(a, y)=1, gcd(x, b)=1, and gcd(x, y)=1

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  - now find **s** suth that $5\,s + 789 * 4 = 1$
    $s = (1 – 3156) / 5 = -631$

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Matrix inversion under mod n

- A square matrix is invertible mod n if and only if its determinant and n are relatively prime
- ex: in real field R

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

In a finite field Z (mod n)? we need to find the inverse for ad-bc (mod n) in order to calculate the inverse of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \equiv (ad - bc)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \pmod{n}$$

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- **Definition:** The ring Z_m consists of
  - The set $Z_m = \{0, 1, 2, \ldots, m\text{-}1\}$
  - Two operations "+ (mod m)" and "× (mod m)" for all $a, b \in Z_m$ such that they satisfy the properties on the next slide

- **Example:** $m = 9$ $Z_9 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

  $6 + 8 = 14 \equiv 5 \pmod 9$

  $6 \times 8 = 48 \equiv 3 \pmod 9$

Properties of the ring Z_m

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  - Multiplicative identity "1": $a \times 1 \equiv a \pmod{m}$
  - The multiplicative inverse of a exists only when $\gcd(a,m) = 1$ and denoted as a^{-1} s.t. $a^{-1} \times a \equiv 1 \pmod{m}$
  - Multiplication is closed i.e. if $a, b \in Z_m$ then $a \times b \in Z_m$
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  - Multiplicative identity "1": $a \times 1 \equiv a \pmod{m}$
  - The multiplicative inverse of a exists only when $\gcd(a,m) = 1$ and denoted as a^{-1} s.t. $a^{-1} \times a \equiv 1 \pmod{m}$ **might or might not exist**
  - Multiplication is closed i.e. if $a, b \in Z_m$ then $a \times b \in Z_m$
  - Multiplication is associative $(a \times b) \times c \equiv a \times (b \times c) \pmod{m}$
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Some remarks on the ring Z_m

- A ring is an Abelian group under addition and an Abelian semigroup under multiplication..

Some remarks on the ring Z_m

- A ring is an Abelian group under addition and an Abelian semigroup under multiplication..
- A semigroup is defined for a set and an associative **binary operator**. No other restrictions are placed on a semigroup; thus a semigroup <u>need not have an **identity**</u> element and its elements <u>need not have **inverses**</u> within the semigroup.

Some remarks on the ring Z_m (cont'd)

✧ Roughly speaking a ring is a mathematical structure in which we can add, subtract, multiply, and even sometimes divide. (A ring in which every element has multiplicative inverse is called a field.)

Some remarks on the ring Z_m (cont'd)

- Roughly speaking a ring is a mathematical structure in which we can add, subtract, multiply, and even sometimes divide. (A ring in which every element has multiplicative inverse is called a field.)
  - **Example:** Is the division 4/15 (mod 26) possible?

    In fact, $4/15 \bmod 26 \equiv 4 \times 15^{-1} \pmod{26}$

    Does $15^{-1} \pmod{26}$ exist ?

    It exists only if $\gcd(15, 26) = 1$.

    $15^{-1} \equiv 7 \pmod{26}$ therefore,

    $4/15 \bmod 26 \equiv 4 \times 7 \equiv 28 \equiv 2 \bmod 26$

Some remarks on the group Z_m *and* Z_m^*

✧ The modulo operation can be applied whenever we want

Some remarks on the group Z_m *and* $Z_m{}^*$

- The modulo operation can be applied whenever we want

in Z_m

$(a + b) \pmod{m} \equiv [(a \pmod{m}) + ((b \bmod m))\ ] \pmod{m}$

Some remarks on the group Z_m *and* Z_m^*

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in Z_m

$$(a + b) \ (\mathrm{mod}\ m) \equiv [(a \ (\mathrm{mod}\ m)) + ((b \ \mathrm{mod}\ m)) \ ] \ (\mathrm{mod}\ m)$$

in Z_m^*

$$(a \times b) \ (\mathrm{mod}\ m) \equiv [(a \ (\mathrm{mod}\ m)) \times ((b \ \mathrm{mod}\ m)) \ ] \ (\mathrm{mod}\ m)$$

$$a^b \ (\mathrm{mod}\ m) \equiv (a \ (\mathrm{mod}\ m))^b \ (\mathrm{mod}\ m)$$

Some remarks on the group Z_m *and* Z_m^*

✧ The modulo operation can be applied whenever we want

in Z_m

$$(a + b) \pmod{m} \equiv [(a \pmod{m}) + ((b \bmod m))] \pmod{m}$$

in Z_m^*

$$(a \times b) \pmod{m} \equiv [(a \pmod{m}) \times ((b \bmod m))] \pmod{m}$$

$$a^b \pmod{m} \equiv (a \pmod{m})^b \pmod{m}$$

Question? $a^b \pmod{m} \stackrel{?}{\equiv} a^{(b \bmod m)} \pmod{m}$

Exponentiation in Z_m

- Example: $3^8 \pmod 7 \equiv ?$

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or

$$3^8 \pmod 7 \equiv 3^4 \times 3^4 \pmod 7 \equiv 3^2 \times 3^2 \times 3^2 \times 3^2 \pmod 7$$

$$\equiv (3^2 \pmod 7)\times(3^2 \pmod 7)\times(3^2 \pmod 7)\times(3^2 \pmod 7)$$

$$\equiv 2 \times 2 \times 2 \times 2 \pmod 7 \equiv 16 \pmod 7 \equiv 2$$

Exponentiation in Z_m

✧ Example: $3^8 \pmod 7 \equiv ?$

$$3^8 \pmod 7 \equiv 6561 \pmod 7 \equiv 2 \text{ since } 6561 \equiv 937 \times 7 + 2$$

or

$$\begin{aligned} 3^8 \pmod 7 &\equiv 3^4 \times 3^4 \pmod 7 \equiv 3^2 \times 3^2 \times 3^2 \times 3^2 \pmod 7 \\ &\equiv (3^2 \pmod 7)\times(3^2 \pmod 7)\times(3^2 \pmod 7)\times(3^2 \pmod 7) \\ &\equiv 2 \times 2 \times 2 \times 2 \pmod 7 \equiv 16 \pmod 7 \equiv 2 \end{aligned}$$

✧ The cyclic group Z_m^* and the modulo arithmetic is of central importance to modern public-key cryptography. In practice, the order of the integers involved in PKC are in the range of $[2^{160}, 2^{1024}]$. Perhaps even larger.

Exponentiation in Z_m (cont'd)

✧ How do we do the exponentiation efficiently?

Exponentiation in Z_m (cont'd)

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- $3^{1234} \pmod{789}$ many ways to do this

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  a. do 1234 times multiplication and then calculate remainder

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- 3^{1234} (mod 789)   many ways to do this

  a. do 1234 times multiplication and then calculate remainder

  b. repeat 1234 times (multiplication by 3 and calculate remainder)

Exponentiation in Z_m (cont'd)

- How do we do the exponentiation efficiently?
- $3^{1234} \pmod{789}$ many ways to do this
  - a. do 1234 times multiplication and then calculate remainder
  - b. repeat 1234 times (multiplication by 3 and calculate remainder)
  - c. repeated $\lfloor \log 1234 \rfloor$ times (square, multiply and calculate remainder)

Exponentiation in Z_m (cont'd)

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- 3^{1234} (mod 789)   many ways to do this

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  ex. first tabulate

3^2 9 (mod 789)	$3^{32} \equiv 459^2 \equiv 18$	$3^{512} \equiv 732^2 \equiv 93$
$3^4 \equiv 9^2 \equiv 81$	$3^{64} \equiv 18^2 \equiv 324$	$3^{1024} \equiv 93^2 \equiv 759$
$3^8 \equiv 81^2 \equiv 249$	$3^{128} \equiv 324^2 \equiv 39$	
$3^{16} \equiv 249^2 \equiv 459$	$3^{256} \equiv 39^2 \equiv 732$	

Exponentiation in Z_m (cont'd)

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- $3^{1234} \pmod{789}$ many ways to do this

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  ex. first tabulate

$3^2 \quad 9 \pmod{789}$	$3^{32} \equiv 459^2 \equiv 18$	$3^{512} \equiv 732^2 \equiv 93$
$3^4 \equiv 9^2 \equiv 81$	$3^{64} \equiv 18^2 \equiv 324$	$3^{1024} \equiv 93^2 \equiv 759$
$3^8 \equiv 81^2 \equiv 249$	$3^{128} \equiv 324^2 \equiv 39$	
$3^{16} \equiv 249^2 \equiv 459$	$3^{256} \equiv 39^2 \equiv 732$	

$1234 = 1024 + 128 + 64 + 16 + 2 \qquad (10011010010)_2$

Exponentiation in Z_m (cont'd)

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- $3^{1234} \pmod{789}$ many ways to do this

  a. do 1234 times multiplication and then calculate remainder

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  ex. first tabulate

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$1234 = 1024 + 128 + 64 + 16 + 2 \qquad (10011010010)_2$

$3^{1234} \equiv 3^{(1024+128+64+16+2)} \equiv (((759 \cdot 39) \cdot 324) \cdot 459) \cdot 9 \equiv 105 \pmod{789}$

Exponentiation in Z_m (cont'd)

calculate $x^y \pmod{m}$ where $y = b_0 \cdot 2^2 + b_1 \cdot 2 + b_2$

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✧ Method 1:

x

Exponentiation in Z_m (cont'd)

calculate $x^y \pmod m$ where $y = b_0 \cdot 2^2 + b_1 \cdot 2 + b_2$

✧ Method 1:

$$x^{b_2}$$

Exponentiation in Z_m (cont'd)

calculate $x^y \pmod m$ where $y = b_0 \cdot 2^2 + b_1 \cdot 2 + b_2$

✧ Method 1:

$$x^{b_2} \xrightarrow{\text{square}} x^2$$

Exponentiation in Z_m (cont'd)

calculate $x^y \pmod{m}$ where $y = b_0 \cdot 2^2 + b_1 \cdot 2 + b_2$

✧ Method 1:

$$\underbrace{x^{b_2} \Rightarrow (x^{b_2})\quad x^2}_{\text{square}}$$

Exponentiation in Z_m (cont'd)

calculate $x^y \pmod m$ where $y = b_0 \cdot 2^2 + b_1 \cdot 2 + b_2$

✧ Method 1:

$$x^{b_2} \Longrightarrow (x^{b_2}) \cdot (x^2)^{b_1}$$

square

Exponentiation in Z_m (cont'd)

calculate $x^y \pmod{m}$ where $y = b_0 \cdot 2^2 + b_1 \cdot 2 + b_2$

✧ Method 1:

$$x^{b_2} \Rightarrow (x^{b_2}) \cdot (x^2)^{b_1} \qquad\qquad x^4$$

square square

Exponentiation in Z_m (cont'd)

calculate $x^y \pmod{m}$ where $y = b_0 \cdot 2^2 + b_1 \cdot 2 + b_2$

✧ Method 1:

$$x^{b_2} \Rightarrow (x^{b_2}) \cdot (x^2)^{b_1} \Rightarrow \left(x^{b_2} \cdot (x^2)^{b_1}\right) \quad x^4$$

square   square

Exponentiation in Z_m (cont'd)

calculate $x^y \pmod m$ where $y = b_0 \cdot 2^2 + b_1 \cdot 2 + b_2$

- Method 1:

$$x^{b_2} \Rightarrow (x^{b_2}) \cdot (x^2)^{b_1} \Rightarrow \left(x^{b_2} \cdot (x^2)^{b_1}\right) \cdot (x^4)^{b_0}$$

square square

Exponentiation in Z_m (cont'd)

calculate $x^y \pmod{m}$ where $y = b_0 \cdot 2^2 + b_1 \cdot 2 + b_2$

✧ Method 1:

$$\underbrace{x^{b_2} \Longrightarrow (x^{b_2}) \cdot (x^2)^{b_1}}_{\text{square}} \Longrightarrow \underbrace{\left(x^{b_2} \cdot (x^2)^{b_1}\right) \cdot (x^4)^{b_0}}_{\text{square}}$$

✧ Method 2:

x

Exponentiation in Z_m (cont'd)

calculate $x^y \pmod{m}$ where $y = b_0 \cdot 2^2 + b_1 \cdot 2 + b_2$

✧ Method 1:

$$\underbrace{x^{b_2} \Longrightarrow (x^{b_2}) \cdot (x^2)}_{\text{square}}{}^{b_1} \Longrightarrow \underbrace{\left(x^{b_2} \cdot (x^2)^{b_1}\right) \cdot (x^4)}_{\text{square}}{}^{b_0}$$

✧ Method 2:

$$x^{b_0}$$

Exponentiation in Z_m (cont'd)

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✧ Method 2:

$$\underbrace{x^{b_0} \qquad (x^{b_0})^2}_{\text{square}}$$

Exponentiation in Z_m (cont'd)

calculate $x^y \pmod m$ where $y = b_0 \cdot 2^2 + b_1 \cdot 2 + b_2$

✧ Method 1:

$$x^{b_2} \Rightarrow (x^{b_2}) \cdot (x^2)^{b_1} \Rightarrow \left(x^{b_2} \cdot (x^2)^{b_1}\right) \cdot (x^4)^{b_0}$$

square square

✧ Method 2:

$$x^{b_0} \Rightarrow (x^{b_0})^2 \cdot x^{b_1}$$

square

Exponentiation in Z_m (cont'd)

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$$\underbrace{x^{b_2} \Longrightarrow (x^{b_2}) \cdot (x^2)^{b_1}}_{\text{square}} \Longrightarrow \underbrace{\left(x^{b_2} \cdot (x^2)^{b_1}\right) \cdot (x^4)^{b_0}}_{\text{square}}$$

✧ Method 2:

$$\underbrace{x^{b_0} \Longrightarrow (x^{b_0})^2}_{\text{square}} \cdot \underbrace{x^{b_1} \qquad (x^{2 \cdot b_0 + b_1})^2}_{\text{square}}$$

Exponentiation in Z_m (cont'd)

calculate $x^y \pmod{m}$ where $y = b_0 \cdot 2^2 + b_1 \cdot 2 + b_2$

- Method 1:

$$\underbrace{x^{b_2} \Longrightarrow (x^{b_2}) \cdot (x^2}_{\text{square}}\underbrace{)^{b_1} \Longrightarrow \left(x^{b_2} \cdot (x^2)^{b_1}\right) \cdot (x^4}_{\text{square}})^{b_0}$$

- Method 2:

$$\underbrace{x^{b_0} \Longrightarrow (x^{b_0})}_{\text{square}}{}^2 \cdot \underbrace{x^{b_1} \Longrightarrow (x^{2 \cdot b_0 + b_1}}_{\text{square}})^2 \cdot x^{b_2}$$

Exponentiation in Z_m (cont'd)

calculate $x^y \pmod m$ where $y = b_0 \cdot 2^2 + b_1 \cdot 2 + b_2$

✧ Method 1:

$$x^{b_2} \Longrightarrow (x^{b_2}) \cdot (x^2)^{b_1} \Longrightarrow \left(x^{b_2} \cdot (x^2)^{b_1}\right) \cdot (x^4)^{b_0}$$

square   square

✧ Method 2:

$$x^{b_0} \Longrightarrow \underline{(x^{b_0})^2} \cdot x^{b_1} \Longrightarrow \underline{(x^{2 \cdot b_0 + b_1})^2} \cdot x^{b_2}$$

square   square

square and **multiply** $\lfloor \log y \rfloor$ times

Exponentiation in Z_m (cont'd)

Method 1:

$$1234 = 1024 + 128 + 64 + 16 + 2 \qquad (10011010010)_2$$

$$3^{1234} \equiv 3^{0+2(1+2(0+2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1))))))))))}$$

$$\equiv 9 \cdot 9^{2(0+2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1)))))))))}$$

$$\equiv 9 \cdot 81^{2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1))))))))}$$

$$\equiv 9 \cdot 249^{2(1+2(0+2(1+2(1+2(0+2(0+2(1)))))))}$$

$$\equiv 9 \cdot 459 \cdot 459^{2(0+2(1+2(1+2(0+2(0+2(1))))))}$$

$$\equiv 9 \cdot 459 \cdot 18^{2(1+2(1+2(0+2(0+2(1)))))}$$

$$\equiv 9 \cdot 459 \cdot 324 \cdot 324^{2(1+2(0+2(0+2(1))))}$$

$$\equiv 9 \cdot 459 \cdot 324 \cdot 39 \cdot 39^{2(0+2(0+2(1)))}$$

$$\equiv 9 \cdot 459 \cdot 324 \cdot 39 \cdot 732^{2(0+2(1))}$$

$$\equiv 9 \cdot 459 \cdot 324 \cdot 39 \cdot 93^{2(1)}$$

$$\equiv 9 \cdot 459 \cdot 324 \cdot 39 \cdot 759 \bmod 789$$

Exponentiation in Z_m (cont'd)

Method 2: $1234 = 1024 + 128 + 64 + 16 + 2 \qquad (10011010010)_2$

$$3^{1234} \equiv 3^{0+2(1+2(0+2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1))))))))))}$$

$$\equiv (3\bullet 3^{2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1))))))))})^2$$

$$\equiv (3\bullet(3^{2(1+2(0+2(1+2(1+2(0+2(0+2(1)))))))})^2)^2$$

$$\equiv (3\bullet((3\bullet 3^{2(0+2(1+2(1+2(0+2(0+2(1))))))})^2)^2)^2$$

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$$\equiv (3\bullet((3\bullet((3\bullet(3\bullet((3^{2(1)})^2)^2)^2)^2)^2)^2)^2)^2$$

$$\equiv (3\bullet((3\bullet((3\bullet(3\bullet(((3^1)^2)^2)^2)^2)^2)^2)^2)^2)^2$$

Chinese Remainder Theorem (CRT)

✧ $\forall\ i \neq j \in \{1,2,\ldots k\}$, $\gcd(r_i, r_j) = 1$, $0 \leq m_i < r_i$

Is there an m that satisfies simultaneously the following set of congruence equations?

$$\begin{aligned} m &\equiv m_1 \pmod{r_1} \\ &\equiv m_2 \pmod{r_2} \\ &\quad \bullet\bullet\bullet \\ &\equiv m_k \pmod{r_k} \end{aligned}$$

Chinese Remainder Theorem (CRT)

- $\forall\ i \neq j \in \{1,2,\ldots k\}$, $\gcd(r_i, r_j) = 1$, $0 \leq m_i < r_i$
  Is there an m that satisfies simultaneously the following set of congruence equations?

$$
\begin{aligned}
m &\equiv m_1 \pmod{r_1} \\
&\equiv m_2 \pmod{r_2} \\
&\quad \cdots \\
&\equiv m_k \pmod{r_k}
\end{aligned}
$$

Chinese Remainder Theorem (CRT)

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&\equiv m_2 \pmod{r_2} \\
&\quad \cdots \\
&\equiv m_k \pmod{r_k}
\end{aligned}
$$

ex: $m \equiv 1 \pmod 3$
$\equiv 2 \pmod 5$
$\equiv 3 \pmod 7$
Note: $\gcd(3,5) = 1$
$\gcd(3,7) = 1$
$\gcd(5,7) = 1$

Chinese Remainder Theorem (CRT)

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&\quad \bullet\bullet\bullet \\
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\end{aligned}
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Note: $\gcd(3,5) = 1$
$\gcd(3,7) = 1$
$\gcd(5,7) = 1$

- 韓信點兵: 三個一數餘一, 五個一數餘二, 七個一數餘三, 請問隊伍中至少有幾名士兵?

Chinese Remainder Theorem (CRT)

- first solution:

Chinese Remainder Theorem (CRT)

- first solution:

$$n = r_1 \, r_2 \cdots r_k$$

Chinese Remainder Theorem (CRT)

- first solution:

  $n = r_1 r_2 \cdots r_k$

  $z_i = n / r_i$

Chinese Remainder Theorem (CRT)

✧ first solution:

$$n = r_1\, r_2 \cdot \cdot \cdot\, r_k$$

$$z_i = n \,/\, r_i$$

$$\exists!\ s_i \in Z^*_{ri} \ \text{ s.t. }\ s_i \cdot z_i \equiv 1 \pmod{r_i}\ (\text{since } \gcd(z_i, r_i) = 1)$$

Chinese Remainder Theorem (CRT)

✧ first solution:

$$n = r_1 \, r_2 \cdot \cdot \cdot r_k$$

$$z_i = n \,/\, r_i$$

$$\exists!\ s_i \in Z^*_{ri} \ \text{ s.t. } \ s_i \cdot z_i \equiv 1 \pmod{r_i} \ (\text{since } \gcd(z_i, r_i) = 1)$$

$$m \equiv \sum_{i=1}^{k} z_i \cdot s_i \cdot m_i \pmod{n}$$

Chinese Remainder Theorem (CRT)

- first solution:

$$n = r_1 r_2 \cdots r_k$$

$$z_i = n / r_i$$

$$\exists!\ s_i \in Z^*_{ri} \ \text{ s.t. } \ s_i \cdot z_i \equiv 1 \ (\text{mod } r_i) \ (\text{since } \gcd(z_i, r_i) = 1)$$

$$m \equiv \sum_{i=1}^{k} z_i \cdot s_i \cdot m_i \ \ (\text{mod } n)$$

- ex: $m_1=1, \ m_2=2, m_3=3$

$r_1=3, \quad r_2=5, \quad r_3=7 \qquad\qquad n = 3 \cdot 5 \cdot 7$

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✧ ex: $m_1=1,\ m_2=2,\ m_3=3$

$r_1=3,\ \ r_2=5,\ \ r_3=7$ $\qquad n = 3 \cdot 5 \cdot 7$

$z_1=35,\ z_2=21,\ z_3=15$

Chinese Remainder Theorem (CRT)

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$$n = r_1 r_2 \cdot \cdot \cdot r_k$$

$$z_i = n / r_i$$

$$\exists!\ s_i \in Z^*_{ri} \text{ s.t. } s_i \cdot z_i \equiv 1 \pmod{r_i} \text{ (since } \gcd(z_i, r_i) = 1)$$

$$m \equiv \sum_{i=1}^{k} z_i \cdot s_i \cdot m_i \pmod{n}$$

✧ ex: $m_1=1,\ m_2=2,\ m_3=3$

$r_1=3,\quad r_2=5,\quad r_3=7 \qquad\qquad n = 3 \cdot 5 \cdot 7$

$z_1=35,\ z_2=21,\ z_3=15$

$s_1=2,\quad s_2=1,\quad s_3=1 \qquad\qquad 35 \cdot 2 + 3\ (-23) = 1$

Chinese Remainder Theorem (CRT)

✧ first solution:

$$n = r_1\, r_2 \cdot \cdot \cdot r_k$$

$$z_i = n \,/\, r_i$$

$$\exists!\ s_i \in Z^*_{ri} \ \text{ s.t. } \ s_i \cdot z_i \equiv 1 \pmod{r_i} \ (\text{since } \gcd(z_i, r_i) = 1)$$

$$m \equiv \sum_{i=1}^{k} z_i \cdot s_i \cdot m_i \pmod{n}$$

✧ ex: $m_1=1,\ m_2=2,\ m_3=3$

$r_1=3,\quad r_2=5,\quad r_3=7 \qquad\qquad n = 3 \cdot 5 \cdot 7$

$z_1=35,\ z_2=21,\ z_3=15$

$s_1=2,\quad s_2=1,\quad s_3=1$

$m \equiv 35\cdot 2\cdot 1 + 21\cdot 1\cdot 2 + 15\cdot 1\cdot 3 \equiv 157 \equiv 52 \pmod{105}$

Chinese Remainder Theorem (CRT)

- first solution:

  $n = r_1 r_2 \cdot \cdot \cdot r_k$

  $z_i = n / r_i$

  $\exists!\ s_i \in Z^*_{ri}$ s.t. $s_i \cdot z_i \equiv 1 \pmod{r_i}$ (since $\gcd(z_i, r_i) = 1$)

  $m \equiv \sum_{i=1}^{k} z_i \cdot s_i \cdot m_i \pmod{n}$

  Unique solution in Z_n?

- ex: $m_1=1,\ m_2=2,\ m_3=3$

  $r_1=3,\quad r_2=5,\quad r_3=7 \qquad\qquad n = 3 \cdot 5 \cdot 7$

  $z_1=35,\ z_2=21,\ z_3=15$

  $s_1=2,\quad s_2=1,\quad s_3=1$

  $m \equiv 35 \cdot 2 \cdot 1 + 21 \cdot 1 \cdot 2 + 15 \cdot 1 \cdot 3 \equiv 157 \equiv 52 \pmod{105}$

Chinese Remainder Theorem (CRT)

✧ Uniqueness:

1. If there exists $m' \in Z_n$ ($\neq m$) also satisfies the previous k congruence relations, then
$$\forall i,\ m'-m \equiv 0 \pmod{r_i}.$$
2. This is equivalent to $\forall i,\ r_i \mid m' - m$
3. $\forall i,j,\ \gcd(r_i, r_j) = 1 \ \Rightarrow\ r_1 r_2 \ldots r_k \mid m' - m$

⇨ $m' = m + k \cdot r_1, r_2 \ldots r_k = m + k \cdot n$

⇨ $m' \notin Z_n$ for all $k \neq 0$

contradiction!

Chinese Remainder Theorem (CRT)

✧ second solution:

$$R_i = r_1 r_2 \cdots r_{i-1}$$

$$\exists!\ t_i \in Z^*_{r_i} \ \text{ s.t. } t_i \cdot R_i \equiv 1 \pmod{r_i} \ (\text{since } \gcd(R_i, r_i) = 1)$$

satisfies the first i-1 congruence relations

$$\begin{cases} \hat{m}_1 = m_1 \\ \hat{m}_i = \hat{m}_{i-1} + R_i \cdot (m_i - \hat{m}_{i-1}) \cdot t_i \pmod{R_{i+1}} \quad i \geq 2 \\ m = \hat{m}_k \end{cases}$$

Note that $\hat{m}_i \equiv m_1 \pmod{r_1}$

$\equiv m_2 \pmod{r_2}$

• • •

$\equiv m_i \pmod{r_i}$

$m_1=1,\ m_2=2,\ m_3=3$

$r_1=3,\ r_2=5,\ r_3=7$

$R_2=3,\ R_3=15,\ R_4=105$

$t_2=2,\ t_3=1$

ex: $\hat{m}_1 \equiv 1$

$\hat{m}_2 \equiv 1+3\cdot(2-1)\cdot 2=7$

$\hat{m} \equiv m_3 \equiv 7+15\cdot(3-7)\cdot 1$

$\equiv -53 \equiv 52 \pmod{105}$

Incremental Calculating By Hand

$$\begin{aligned} m &\equiv \mathbf{1} \pmod{3} \\ &\equiv \mathbf{2} \pmod{5} \\ &\equiv \mathbf{3} \pmod{7} \end{aligned}$$

Incremental Calculating By Hand

$$\begin{aligned} m &\equiv 1 \pmod{3} \\ &\equiv 2 \pmod{5} \\ &\equiv 3 \pmod{7} \end{aligned}$$

① $\hat{m}_1 \equiv 1 \pmod{3}$ … satisfying the 1st eq.

Incremental Calculating By Hand

$$\begin{aligned} m &\equiv 1 \pmod 3 \\ &\equiv 2 \pmod 5 \\ &\equiv 3 \pmod 7 \end{aligned} \qquad\qquad \begin{aligned} m &\equiv 1 \pmod 3 \\ &\equiv 2 \pmod 5 \end{aligned}$$

① $\hat{m}_1 \equiv 1 \pmod 3$ … satisfying the 1st eq.

Incremental Calculating By Hand

$$\begin{aligned} m &\equiv 1 \pmod 3) \\ &\equiv 2 \pmod 5) \\ &\equiv 3 \pmod 7) \end{aligned} \qquad \begin{aligned} m &\equiv 1 \pmod 3) \\ &\equiv 2 \pmod 5) \end{aligned}$$

① $\hat{m}_1 \equiv 1 \pmod 3)$ … satisfying the 1st eq.

② $3 \cdot (-3) + 5 \cdot 2 = 1$

Incremental Calculating By Hand

$m \equiv 1 \pmod{3}$
$\equiv 2 \pmod{5}$
$\equiv 3 \pmod{7}$

$m \equiv 1 \pmod{3}$
$\equiv 2 \pmod{5}$

① $\hat{m}_1 \equiv 1 \pmod{3}$ … satisfying the 1st eq.

inverse of 3 (mod 5)

② $3 \cdot (-3) + 5 \cdot 2 = 1$

Incremental Calculating By Hand

$m \equiv \mathbf{1} \pmod 3$
$\equiv \mathbf{2} \pmod 5$
$\equiv \mathbf{3} \pmod 7$

$m \equiv \mathbf{1} \pmod 3$
$\equiv \mathbf{2} \pmod 5$

① $\hat{m}_1 \equiv \mathbf{1} \pmod 3$ … satisfying the 1st eq.

② $\underbrace{3 \cdot (-3)}_{\text{inverse of 3 (mod 5)}} + \underbrace{5 \cdot 2}_{\text{inverse of 5 (mod 3)}} = 1$

inverse of 3 (mod 5)

inverse of 5 (mod 3)

Incremental Calculating By Hand

$$m \equiv 1 \pmod 3$$
$$\equiv 2 \pmod 5$$
$$\equiv 3 \pmod 7$$

$$m \equiv 1 \pmod 3$$
$$\equiv 2 \pmod 5$$

① $\hat{m}_1 \equiv 1 \pmod 3$ … satisfying the 1st eq.

② $3 \cdot (-3) + 5 \cdot 2 = 1$

inverse of 3 (mod 5)

inverse of 5 (mod 3)

③ $\hat{m}_2 \equiv 2 \cdot 3 \cdot (-3) + \underset{\hat{m}_1}{1} \cdot 5 \cdot 2$

Incremental Calculating By Hand

$m \equiv 1 \pmod 3$
$\equiv 2 \pmod 5$
$\equiv 3 \pmod 7$

$m \equiv 1 \pmod 3$
$\equiv 2 \pmod 5$

① $\hat{m}_1 \equiv 1 \pmod 3$ … satisfying the 1st eq.

inverse of 3 (mod 5)

② $3 \cdot (-3) + 5 \cdot 2 = 1$

inverse of 5 (mod 3)

③ $\hat{m}_2 \equiv \underbrace{2}_{m_2} \cdot 3 \cdot (-3) + \underbrace{1}_{\hat{m}_1} \cdot 5 \cdot 2$

Incremental Calculating By Hand

$$m \equiv 1 \pmod 3 \qquad m \equiv 1 \pmod 3$$
$$\equiv 2 \pmod 5 \qquad \equiv 2 \pmod 5$$
$$\equiv 3 \pmod 7$$

① $\hat{m}_1 \equiv 1 \pmod 3$ … satisfying the 1st eq.

② $3 \cdot (-3) + 5 \cdot 2 = 1$

③ $\hat{m}_2 \equiv 2 \cdot 3 \cdot (-3) + 1 \cdot 5 \cdot 2 \equiv -8 \equiv 7 \pmod{15}$ …. satisfying first 2 eqs.

Incremental Calculating By Hand

$$\begin{aligned} m &\equiv 1 \pmod 3) \\ &\equiv 2 \pmod 5) \\ &\equiv 3 \pmod 7) \end{aligned}$$

$$\begin{aligned} m &\equiv 7 \pmod{15} \\ &\equiv 3 \pmod 7 \end{aligned}$$

① $\hat{m}_1 \equiv 1 \pmod 3$ … satisfying the 1st eq.

② $3 \cdot (-3) + 5 \cdot 2 = 1$

③ $\hat{m}_2 \equiv 2 \cdot 3 \cdot (-3) + 1 \cdot 5 \cdot 2 \equiv -8 \equiv 7 \pmod{15}$ …. satisfying first 2 eqs.

Incremental Calculating By Hand

$$m \equiv 1 \pmod 3$$
$$\equiv 2 \pmod 5$$
$$\equiv 3 \pmod 7$$

$$m \equiv 7 \pmod{15}$$
$$\equiv 3 \pmod 7$$

① $\hat{m}_1 \equiv 1 \pmod 3$ … satisfying the 1st eq.

② $3 \cdot (-3) + 5 \cdot 2 = 1$

③ $\hat{m}_2 \equiv 2 \cdot 3 \cdot (-3) + 1 \cdot 5 \cdot 2 \equiv -8 \equiv 7 \pmod{15}$ …. satisfying first 2 eqs.

④ $15 \cdot 1 + 7 \cdot (-2) = 1$

Incremental Calculating By Hand

$m \equiv 1 \pmod 3$
$\equiv 2 \pmod 5$
$\equiv 3 \pmod 7$

$m \equiv 7 \pmod{15}$
$\equiv 3 \pmod 7$

① $\hat{m}_1 \equiv 1 \pmod 3$ … satisfying the 1st eq.

② $3 \cdot (-3) + 5 \cdot 2 = 1$

③ $\hat{m}_2 \equiv 2 \cdot 3 \cdot (-3) + 1 \cdot 5 \cdot 2 \equiv -8 \equiv 7 \pmod{15}$ …. satisfying first 2 eqs.

④ $15 \cdot 1 + 7 \cdot (-2) = 1$

inverse of 15 (mod 7)

inverse of 7 (mod 15)

Incremental Calculating By Hand

$m \equiv 1 \pmod{3}$
$\quad \equiv 2 \pmod{5}$
$\quad \equiv 3 \pmod{7}$

$m \equiv 7 \pmod{15}$
$\quad \equiv 3 \pmod{7}$

① $\hat{m}_1 \equiv 1 \pmod{3}$ … satisfying the 1st eq.

② $3 \cdot (-3) + 5 \cdot 2 = 1$

③ $\hat{m}_2 \equiv 2 \cdot 3 \cdot (-3) + 1 \cdot 5 \cdot 2 \equiv -8 \equiv 7 \pmod{15}$ …. satisfying first 2 eqs.

inverse of 15 (mod 7)

④ $15 \cdot 1 + 7 \cdot (-2) = 1$

inverse of 7 (mod 15)

⑤ $\hat{m}_3 \equiv 3 \cdot 15 \cdot 1 + 7 \cdot 7 \cdot (-2)$

$\hat{m}_2$

Incremental Calculating By Hand

$m \equiv 1 \pmod{3}$
$\equiv 2 \pmod{5}$
$\equiv 3 \pmod{7}$

$m \equiv 7 \pmod{15}$
$\equiv 3 \pmod{7}$

① $\hat{m}_1 \equiv 1 \pmod{3}$ … satisfying the 1st eq.

② $3 \cdot (-3) + 5 \cdot 2 = 1$

③ $\hat{m}_2 \equiv 2 \cdot 3 \cdot (-3) + 1 \cdot 5 \cdot 2 \equiv -8 \equiv 7 \pmod{15}$ …. satisfying first 2 eqs.

inverse of 15 (mod 7)

④ $15 \cdot 1 + 7 \cdot (-2) = 1$

inverse of 7 (mod 15)

⑤ $\hat{m}_3 \equiv 3 \cdot 15 \cdot 1 + 7 \cdot 7 \cdot (-2)$

m_3 $\hat{m}_2$

Incremental Calculating By Hand

$$m \equiv 1 \pmod 3$$
$$\equiv 2 \pmod 5$$
$$\equiv 3 \pmod 7$$

① $\hat{m}_1 \equiv 1 \pmod 3$ … satisfying the 1st eq.

② $3 \cdot (-3) + 5 \cdot 2 = 1$

③ $\hat{m}_2 \equiv 2 \cdot 3 \cdot (-3) + 1 \cdot 5 \cdot 2 \equiv -8 \equiv 7 \pmod{15}$ …. satisfying first 2 eqs.

④ $15 \cdot 1 + 7 \cdot (-2) = 1$

⑤ $\hat{m}_3 \equiv 3 \cdot 15 \cdot 1 + 7 \cdot 7 \cdot (-2) \equiv -53 \equiv \mathbf{52} \pmod{105}$ … satisfying all 3 eqs.

Chinese Remainder Theorem (CRT)

- special case:

$$x \equiv m \pmod{r_1} \equiv m \pmod{r_2} \cdots \equiv m_n \pmod{r_n} \Rightarrow x \equiv m \pmod{r_1 r_2 \cdots r_n}$$

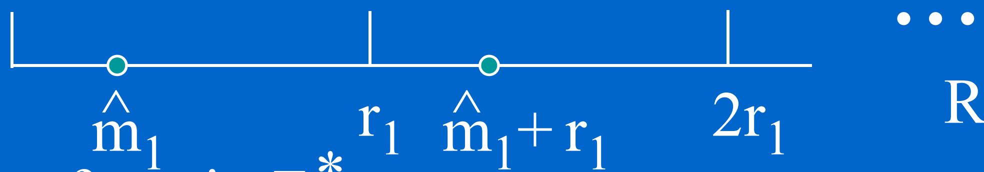
- insight of the second solution:

every step satisfies one more equation

step 1

$x \equiv m_1 \pmod{r_1}$

let $\hat{m}_1 = m_1$



$R_2 = r_1$

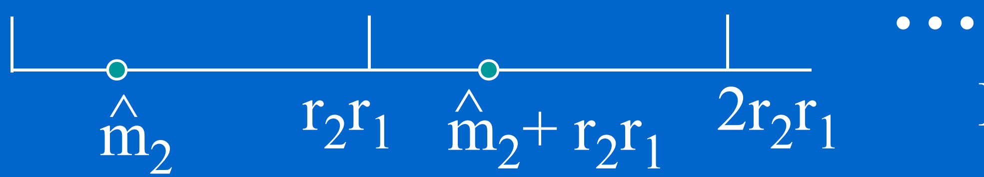
m_1 is the only solution for x in $Z^*_{R_2}$

general solution of x must be $\hat{m}_1 + k R_2$ for some k

step 2

$x \equiv m_1 \pmod{r_1}$

$\quad \equiv m_2 \pmod{r_2}$



$R_3 = r_2 r_1$

let $\hat{m}_2 \equiv \hat{m}_1 + k^* R_2 \pmod{R_3}$ where $k^* = t_2(m_2 - \hat{m}_1)$ and $t_2 R_2 \equiv 1 \pmod{r_2}$

m_2 is the only solution for x in $Z^*_{R_3}$

general solution of x must be $\hat{m}_2 + k R_3$ for some k

Chinese Remainder Theorem (CRT)

- Applications: solve $x^2 \equiv 1 \pmod{35}$
  - $35 = 5 \cdot 7$
  - x^* satisfies $f(x^*) \equiv 0 \pmod{35}$ $\Leftrightarrow$
    x^* satisfies both $f(x^*) \equiv 0 \pmod{5}$ and $f(x^*) \equiv 0 \pmod{7}$

  Proof:

  $(\Leftarrow)$

  $p \mid f(x^*)$, $q \mid f(x^*)$, and $\gcd(p,q)=1$ imply that
  $p \cdot q \mid f(x^*)$ i.e. $f(x^*) \equiv 0 \pmod{p \cdot q}$

  $(\Rightarrow)$

  $f(x^*) = k \cdot p \cdot q$ implies that
  $f(x^*) = (k \cdot p) \cdot q = (k \cdot q) \cdot p$ i.e. $f(x^*) \equiv 0 \pmod{p}$
  $\equiv 0 \pmod{q}$

Chinese Remainder Theorem (CRT)

- since 5 and 7 are prime, we can solve

  $$x^2 \equiv 1 \pmod 5 \text{ and } x^2 \equiv 1 \pmod 7$$

  far more easily than $x^2 \equiv 1 \pmod{35}$

  Why?

  - $x^2 \equiv 1 \pmod 5$ has exactly two solutions: $x \equiv \pm 1 \pmod 5$
  - $x^2 \equiv 1 \pmod 7$ has exactly two solutions: $x \equiv \pm 1 \pmod 7$
- put them together and use CRT, there are four solutions
  - $x \equiv 1 \pmod 5 \equiv 1 \pmod 7 \Rightarrow x \equiv 1 \pmod{35}$
  - $x \equiv 1 \pmod 5 \equiv 6 \pmod 7 \Rightarrow x \equiv 6 \pmod{35}$
  - $x \equiv 4 \pmod 5 \equiv 1 \pmod 7 \Rightarrow x \equiv 29 \pmod{35}$
  - $x \equiv 4 \pmod 5 \equiv 6 \pmod 7 \Rightarrow x \equiv 34 \pmod{35}$

Matlab tools

	format rat format long
matrix inverse	inv(A)
matrix determinant	det(A)
p = q d + r	r = mod(p, d) or r = rem(p, d)
	q = floor(p / d)
	g = gcd(a, b)
g = a s + b t	[g, s, t] = gcd(a, b)
factoring	factor(N)
prime numbers < N	primes(N)
test prime	isprime(p)
mod exponentiation *	powermod(a,b,n)
find primitive root *	primitiveroot(p)
crt *	crt([a_1 a_2 a_3...], [m_1 m_2 m_3...])
$\phi(N)$ *	eulerphi(N)

Field

- Field: a set that has the operation of addition, multiplication, subtraction, and division by nonzero elements. Also, the associative, commutative, and distributive laws hold.
- Ex. Real numbers, complex numbers, rational numbers, integers mod a prime are fields
- Ex. Integers, 2×2 matrices with real entries are not fields
- Ex. $GF(4) = \{0, 1, \omega, \omega^2\}$
  - $0 + x = x$
  - $x + x = 0$
  - $1 \cdot x = x$
  - $\omega + 1 = \omega^2$

- Addition and multiplication are commutative and associative, and the distributive law $x(y+z)=xy+xz$ holds for all x, y, z
- $x^3 = 1$ for all nonzero elements

Galois Field

- ✧ Galois Field: A field with finite element, finite field
- ✧ For every power p^n of a prime, there is exactly one finite field with p^n elements, $GF(p^n)$, and these are the only finite fields.
- ✧ For $n > 1$, {integers (mod p^n)} do not form a field.
  - ✶ Ex. $p \cdot x \equiv 1 \pmod{p^n}$ does not have a solution
    (i.e. p does not have multiplicative inverse)

How to construct a GF(p^n)?

- Def: $Z_2[X]$: the set of polynomials whose coefficients are integers mod 2
  - ex. 0, 1, $1+X^3+X^6$…
  - add/subtract/multiply/divide/Euclidean Algorithm: process all coefficients mod 2
    - $(1+X^2+X^4) + (X+X^2) = 1+X+X^4$ bitwise XOR
    - $(1+X+X^3)(1+X) = 1+X^2+X^3+X^4$
    - $X^4+X^3+1 = (X^2+1)(X^2+X+1) + X$ long division
      can be written as
      $X^4+X^3+1 \equiv X \pmod{X^2+X+1}$

How to construct $GF(2^n)$?

- Define $Z_2[X]$ (mod X^2+X+1) to be $\{0, 1, X, X+1\}$
  - addition, subtraction, multiplication are done mod X^2+X+1
  - $f(X) \equiv g(X)$ (mod X^2+X+1)
    - if $f(X)$ and $g(X)$ have the same remainder when divided by X^2+X+1
    - or equivalently $\exists$ $h(X)$ such that $f(X) - g(X) = (X^2+X+1)\, h(X)$
    - ex. $X \cdot X = X^2 \equiv X+1$ (mod X^2+X+1)
  - if we replace X by ω, we can get the same GF(4) as before
  - the modulus polynomial X^2+X+1 should be irreducible

Irreducible: polynomial does not factor into polynomials of lower degree with mod 2 arithmetic
ex. X^2+1 is not irreducible since $X^2+1 = (X+1)(X+1)$

How to construct GF(p^n)?

- $Z_p[X]$ is the set of polynomials with coefficients mod p
- Choose P(X) to be any one irreducible polynomial mod p of degree n (other irreducible P(X)'s would result to isomorphisms)
- Let GF(p^n) be $Z_p[X]$ mod P(X)

- An element in $Z_p[X]$ mod P(X) must be of the form

$$a_0 + a_1 X + \ldots + a_{n-1} X^{n-1}$$

each a_i are integers mod p, and have p choices, hence there are p^n possible elements in GF(p^n)
- multiplicative inverse of any element in GF(p^n) can be found using extended Euclidean algorithm (over polynomial)

GF(2^8)

- AES (Rijndael) uses GF(2^8) with irreducible polynomial $X^8 + X^4 + X^3 + X + 1$
- each element is represented as
  $b_7 X^7 + b_6 X^6 + b_5 X^5 + b_4 X^4 + b_3 X^3 + b_2 X^2 + b_1 X + b_0$
  each b_i is either 0 or 1
- elements of GF(2^8) can be represented as 8-bit bytes
  $b_7b_6b_5b_4b_3b_2b_1b_0$
- mod 2 operations can be implemented by XOR in H/W

GF(p^n)

- Definition of generating polynomial g(X) is parallel to the generator in Z_p:
  - every element in GF(p^n) (except 0) can be expressed as a power of g(X)
  - the smallest exponent k such that $g(X)^k \equiv 1$ is $p^n - 1$
- Discrete log problem in GF(p^n):
  - given h(X), find an integer k such that
    $$h(X) \equiv g(X)^k \pmod{P(X)}$$
  - believed to be very hard in most situations

Recursive GCD

```
int gcd(int p, int q) // assume p >= q
{
    int ans;

    if (p % q == 0)
        ans = q;
    else
        ans = gcd(q, p % q);

    return ans;
}
```

Recursive GCD

```
int gcd(int p, int q) // assume p >= q
{
    int ans;

    if (p % q == 0)
        ans = q;
    else
        ans = gcd(q, p % q);

    return ans;
}
```

```
int gcd(int p, int q)
{
    int r = p%q;
    if (r == 0)
        return q;
    return gcd(q, r);
}
```

Recursive Extended GCD

- Given $a>b\geq 0$, find $g=GCD(a,b)$ and x, y s.t. $a\,x + b\,y = g$ where $|x|\leq b+1$ and $|y|\leq a+1$
- Let $a = q\,b + r$, $b>r\geq 0 \Rightarrow (q\,b + r)\,x + b\,y = g$

$$\Rightarrow b\,(q\,x + y) + r\,x = g$$

$$\Rightarrow b\,y' + r\,x = g, \text{ where } y' = q\,x + y$$

- This means that if we can find y' and x satisfying $b\,y' + (a\%b)\,x = g$ then x and $y = y' - q\,x = y' - (a/b)\,x$ satisfies $a\,x + b\,y = g$
  Note that in this way r will eventually be 0

```
void extgcd(int a, int b, int *g, int *x, int *y) { // a > b >=0
    if (b == 0)
        *g = a, *x = 1, *y = 0;
    else {
        extgcd(b, a%b, g, y, x);
        *y = *y - (a/b)*(*x);
    }
}
```