RSA Cryptosystem

密碼學與應用

海洋大學資訊工程系
丁培毅

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(In $m * 8 * 8^{-1}$, the brace over $8 * 8^{-1}$ links it to the 1 in the line above.)

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8 is the public key
m * 8 is the ciphertext
8^{-1} is the private key (if nobody can derive this from the public key, then this system is secure)

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$\{a_1, a_2, \ldots a_n\}$ such that $a_i > \sum_{j=0}^{i-1} a_j$ e.g. 1, 3, 5, 10, 20, 40

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  3. Every possible subset sum is unique

  pf: given x, assume $x = \underset{j\in S}{} a_j = \sum_{j\in T} a_j$, where S T, assume $\max_{j\in S}\{a_j\}$ $_{j\in T}$ $\max\{a_j\}$ ….

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- choose a number ***b*** in Z_p^*, e.g. $p = 101$, ***b*** $= 23$, and convert the **super-increasing sequence** to a **normal knapsack sequence** $B=\{b_1, b_2, \ldots, b_n\}$ where $b_i \equiv a_i \cdot b \pmod{p}$

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- **G**iven a number d, finding a subset $\{b_j\} \subseteq B$ s.t.

  $$d = \sum_j b_j \pmod{p}$$

  is an NP-complete problem, e.g. 94 = 11 + 14 + 69

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  - recovered message is $(111100)_2 = (60)_{10}$

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both functions are candidates for **trapdoor one way function**

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  $y \quad x^e \pmod{n}$, where $\gcd(e, \quad (n)) = 1$

  Why don't we take (e^{-1})-th power of y?

  where $e^{-1} \cdot e \quad 1 \pmod{\quad (n)}$

  e.g. $n = 11 \cdot 13 = 143$, $e = 7$

  $\phi(n) = 10 \cdot 12 = 120$, $e^{-1} = 103$

Trouble: How do we know $\phi(n)$?

- Solving square root of y modulo n is difficult!!!

  $y \quad x^2 \pmod{n}$

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$$\Rightarrow \forall k, \forall x \neq r \cdot p,\ x^{k\phi(n)} \equiv 1 \pmod{p}, \forall x \neq s \cdot q,\ x^{k\phi(n)} \equiv 1 \pmod{q}$$

CRT

$$\Rightarrow \forall k, \forall x,\ x^{k\phi(n)+1} \equiv x \pmod{p},\ x^{k\phi(n)+1} \equiv x \pmod{q}$$

$$\Rightarrow \forall k, \forall x,\ x^{k\phi(n)+1} \equiv x \pmod{n}$$

- $\gcd(e, \phi(n)) = 1 \quad \Rightarrow$ inverse of e (mod $\phi(n)$) exists
  $\Rightarrow$ let d be the inverse s.t. $e \cdot d \equiv 1 \pmod{\phi(n)}$
- $\forall x_1, x_2 \in Z_n$ if $x_1^e \equiv x_2^e \pmod{n}$

$$\Rightarrow (x_1^e)^d \equiv (x_2^e)^d \pmod{n}$$

$$\Rightarrow (x_1)^{1+k\,\phi(n)} \equiv (x_2)^{1+k\,\phi(n)} \pmod{n}$$

$$\Rightarrow x_1 \equiv x_2 \pmod{n}$$

RSA Function is a Permutation

- RSA function is a permutation: (1-1 and onto, bijective)
- Goal: "$\forall x_1, x_2 \in Z_n$ if $x_1^e \equiv x_2^e \pmod n$ then $x_1 = x_2$"

$\forall x \neq r\cdot p,\ x^{p-1} \equiv 1 \pmod p$, $\forall x \neq s\cdot q,\ x^{q-1} \equiv 1 \pmod q$

$\Rightarrow \forall k, \forall x \neq r\cdot p,\ x^{k\phi(n)} \equiv 1 \pmod p, \forall x \neq s\cdot q,\ x^{k\phi(n)} \equiv 1 \pmod q$

CRT $\Rightarrow \forall k, \forall x,\ x^{k\phi(n)+1} \equiv x \pmod p$, $x^{k\phi(n)+1} \equiv x \pmod q$

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Note: Euler Thm is valid only when $x \in Z_n^*$

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- In constrained devices such as smart cards, cellular phones and PDAs, it is hard to store, communicate keys or handle operations involving large integers

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Very likely to be relatively prime with (p-1)(q-1)

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from gmpy2 import mpz, next_prime, invert, powmod

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phi = (p-1)*(q-1)                      # 528370842090492127360
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ciphertext = powmod(plaintext, e, n)      # 479679342785929350234
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d = invert(e, phi)                    # 139387972146660337833

plaintext = 101
ciphertext = powmod(plaintext, e, n)      # 479679342785929350234
decrypted = powmod(ciphertext, d, n)      # 101
```

Rabin Cryptosystem (1/3)

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- Bob computes the four square roots $\pm m_1$, $\pm m_2$ using CRT, one of them satisfying the fixed message format is the recovered message

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  - In order to let the Rabin function have inverse, it is necessary to make the Rabin function a permutation, ie. 1-1 and onto. Therefore, the number of elements in the domain of the Rabin function should also be $(p-1)(q-1)/4$ for $n=p\cdot q$. There are 4 possible numbers with their square equal to y, and we have to make 3 of them illegal.

Number of Quadratic Residues

- For a prime modulus p: number of QR_p's in Z_p^* is $(p-1)/2$

  pf: find a primitive g, at least $\{g^2, g^4, \ldots g^{p-1}\}$ are QR_p's
  assume there are $(p+1)/2$ QRs,
  since there are exactly two square roots of a QR modulo p
  there are $p+1$ square roots for these $(p+1)/2$ QRs, i.e. there must be at least two pairs of square roots are the same (pigeon-hole), i.e. two out of these $(p+1)/2$ QRs are the same, contradiction

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  there are p+1 square roots for these (p+1)/2 QRs, i.e. there must be at least two pairs of square roots are the same (pigeon-hole), i.e. two out of these (p+1)/2 QRs are the same, contradiction

- For a composite modulus p·q: number of QR_n's in $Z_{p \cdot q}^*$ is (p-1)(q-1)/4

  pf: find a common primitive in Z_p^* and Z_q^* g, at least $\{g^2, g^4, \ldots, g^{p-1} \ldots, g^{q-1} \ldots, g^{\lambda(n)}\}$ are QR_n's, where $\lambda(n) = \mathrm{lcm}(p-1,q-1)$ can be as large as (p-1)(q-1)/2, this set has (p-1)(q-1)/4 distinct elements
  assume there are (p-1)(q-1)/4+1 QR_n's in Z_n^*, since there are four square roots of a QR modulo p·q, these QR_n's have (p-1)(q-1)+4 square roots in total. There must be some repeated elements in this QR_n, therefore, there are at most (p-1)(q-1)/4 QR_n's in Z_n^*

Matlab examples

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- maple('m1:= chrem([r1, r2], [p, q])') % 37044403025442646623512 19

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- maple('m1:= chrem([r1, r2], [p, q])') % 3704440302544264662351219
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- maple('m2:= chrem([-r1, r2], [p, q])') % 704111114221417111030000
- maple('m3:= chrem([r1, -r2], [p, q])') % 52132813183421605542840 41

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- maple('m2:= chrem([-r1, r2], [p, q])') % 704111114221417110300000
- maple('m3:= chrem([r1, -r2], [p, q])') % 5213281318342160554284041
- maple('m4:= chrem([-r1, -r2], [p, q])') % 1579252127220037602962822

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  - $\Rightarrow$
    - given a quadratic residue y if you can find the four square roots $\pm x_1$ and $\pm x_2$ for y in polynomial time
    - you can factor n by trying $\gcd(x_1 - x_2, n)$ and $\gcd(x_1 + x_2, n)$

Basic Factoring Principle (1/4)

- Let n be an integer and suppose there exist integers x and y with $x^2 \equiv y^2 \pmod{n}$, but $x \neq \pm y \pmod{n}$. Then ➊ n is composite, ➋ both $\gcd(x-y, n)$ and $\gcd(x+y, n)$ are nontrivial factors of n.

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Case 1 and 2 implies that $1 < d < n$

i.e. d must be a nontrivial factor of n

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- $x^2 \equiv y^2 \pmod{p}$ implies $x \equiv \pm y \pmod{p}$ since $p \mid (x+y)(x-y)$ implies $p \mid (x+y)$ or $p \mid (x-y)$,
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  - as long as we have z (where $z \not\equiv \pm y$), we can factor n into gcd(y-z, n) and gcd(y+z, n)

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- Compute $b_1 \equiv b_0^2 \pmod{n}$
  if $b_1 \equiv 1 \pmod{n}$, stop, $\gcd(b_0-1, n)$ is a factor of n
  if $b_1 \equiv -1 \pmod{n}$, stop, n is probably prime

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Miller-Rabin Test

Is ***n*** a composite number?

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n will pass Fermat test
n is called pseudo prime with respect to base a

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$n-1 = 2^k \cdot m$

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Consider 4 possible cases:

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there is no chance to use

Basic Factoring Principle, **abort**

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Basic Factoring Principle applied, **composite**

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there is no chance to use
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④ ①, ②, and ③ are not true,
$b_k \equiv a^{n-1} \pmod n$
if $b_k \not\equiv 1 \pmod n$ n is **composite**
since if n is prime, $b_k \equiv 1 \pmod n$
($b_k \equiv 1 \pmod n$ is covered by ②)

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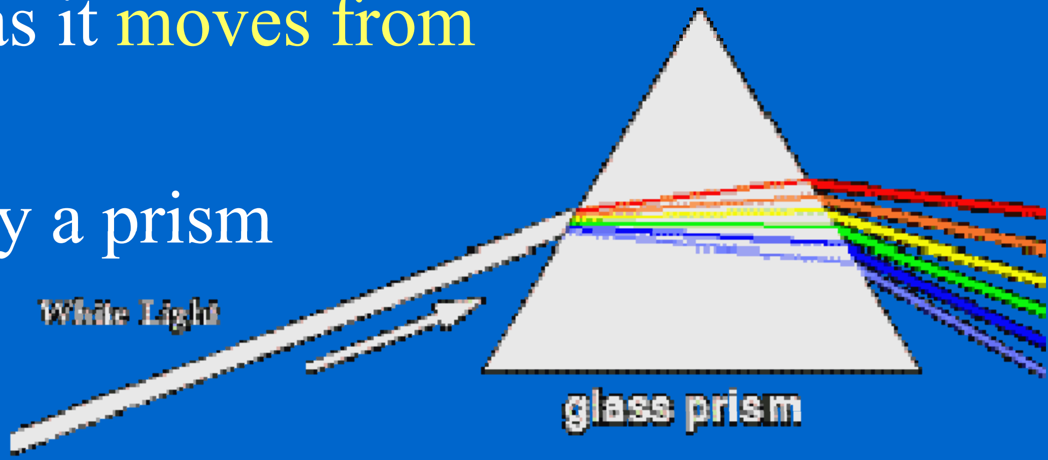

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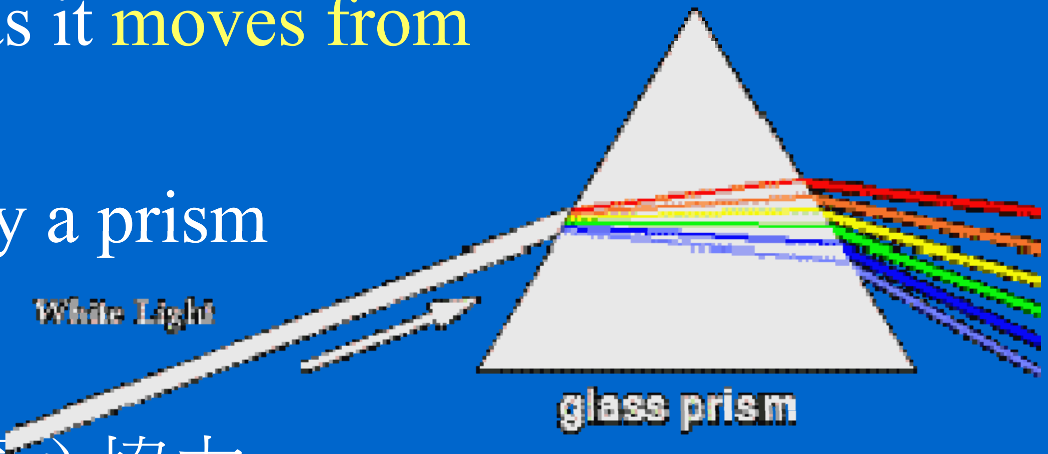

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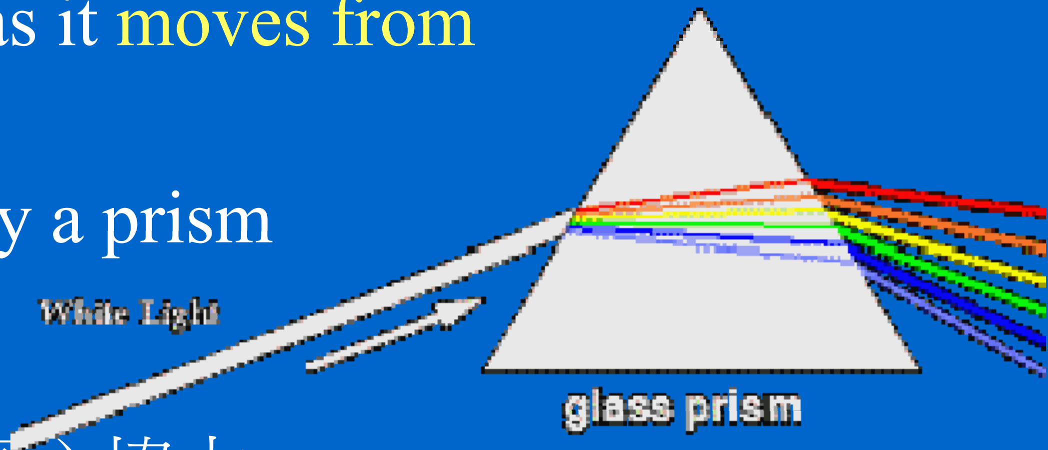

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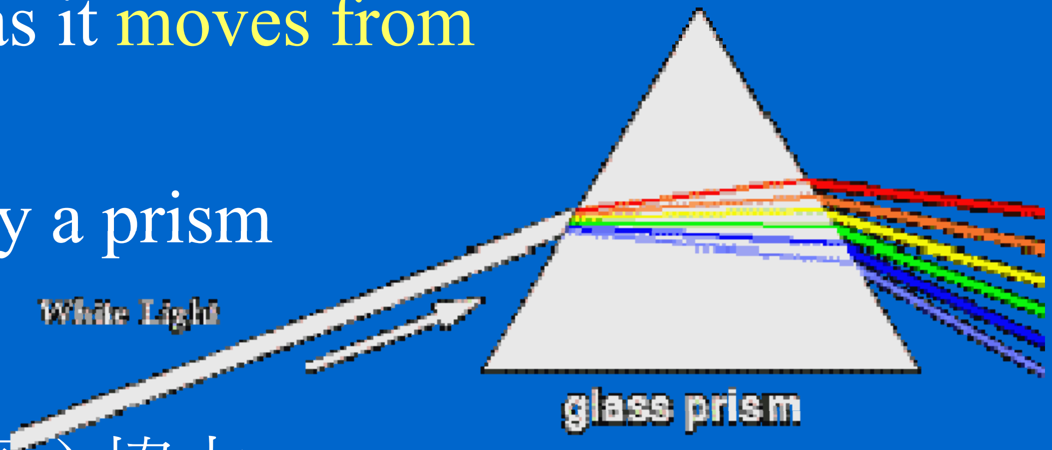

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	2^2	2^3	2^4	2^5	2^6	2^7	2^8
mod 11	4	8	**5**	**10**	**9**	**7**	**3**
mod 13	4	8	**3**	**6**	**12**	**11**	**9**

When/How does Basic Factoring Principle work in M-R test?

- When:
  - explicitly: $b_{i-1} \neq \pm 1 \pmod{n}$ and $b_i \equiv b_{i-1}^2 \equiv 1 \pmod{n}$

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  - catching the moment that $b_0, b_1, \ldots$ behave differently while taking square in (mod p) component and (mod q) component

Miller-Rabin Test Example

- e.g. $n = 561$

  $n-1 = 560 = 16 \cdot 35 = 2^4 \cdot 35$

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A Carmichael number: pass the Fermat test for all bases

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Pseudo Prime and Strong Pseudo Prime

✧ If $\boldsymbol{n}$ is not a prime but satisfies $\boldsymbol{a}^{\boldsymbol{n}-1} \equiv 1 \pmod{\boldsymbol{n}}$ we say that $\boldsymbol{n}$ is a pseudo prime number for base $\boldsymbol{a}$.

Pseudo Prime and Strong Pseudo Prime

- If n is not a prime but satisfies $a^{n-1} \equiv 1 \pmod{n}$ we say that n is a pseudo prime number for base a.
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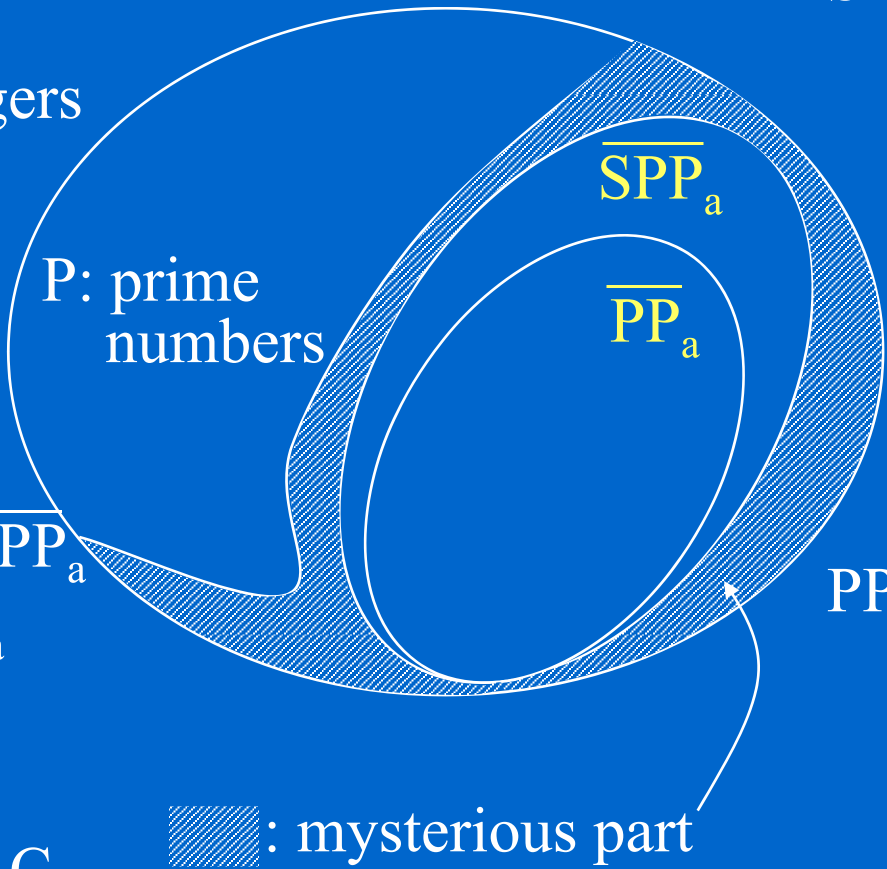
- Up to 10^{10}, there are 455052511 primes, there are 14884 pseudo prime numbers for the base 2, and 3291 strong pseudo prime numbers for the base 2

Fermat and Miller-Rabin Test

- Both of these two tests are for identifying subsets of composite numbers

I: integers



SPP_a: strong pseudo prime numbers for base a, the set of composite n where M-T test says 'probably prime'

C: composite numbers

$I = P \cup C$

$C = SPP_a \cup \overline{SPP_a}$
$\quad = PP_a \cup \overline{PP_a}$

$SPP_a \subset PP_a$

$\overline{PP_a} \subset \overline{SPP_a} \subset C$

PP_a: pseudo prime numbers for base a, the set of composite n where $a^{n-1} \equiv 1 \pmod{n}$

(hatched area): mysterious part
not prime, but cannot be identified as composite

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- However, these tests do not necessarily give you the factors of n in order to tell you that n is a composite number. The factors of n, i.e. p or q, are certainly a kind of witness about the fact that n is composite.
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- A composite number will be factored out by the M-R test only if it is a pseudo prime but it is not a strong pseudo prime number.

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- Randomized algorithms like Rabin-Miller are far more efficient than the IIT algorithm, so we'll keep using those

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- maple('a:=nextprime(189734535789)')

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$$\phi(n) \mid d \cdot e-1,\ d \cdot e-1 \text{ is a universal exponent}$$

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  let r=2^5*r1; r1=32763658052497677873

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- Exponent factorization even if *r* is valid for one *a*, you can still try the above procedure

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If $n=p\cdot q$, $p-1$ and $q-1$ both have small factors that are less than B, then $\gcd(b-1,n)=n$, (useless) however, $b \equiv a^{B!} \equiv 1 \pmod{n}$ and we can use the Universal exponent method

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- Best records: p-1: 34 digits (113 bits), ECM: 47 digits (143 bits)

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- the other factor is 3837523/1093 = 3511

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19095	2	0	1	0	1	1	0	1
1964	0	2	0	0	0	3	0	0
17078	6	2	0	0	1	0	0	0
8077	1	0	0	0	0	0	0	1
3397	5	0	1	0	0	2	0	0
14262	0	0	2	2	0	1	0	0

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- How do we construct these useful relations systematically?
- Properties of these relations:
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19095	2	0	1	0	1	1	0	1
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- Looking for those $x^2 \equiv y^2$ but $x \neq \pm y$

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Next challenge
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  - Note: There are four roots to the equation $b^2 \equiv 1 \pmod n$, ± 1 are two of them, all satisfy $(b+1)(b-1) = k\cdot n = k\cdot p\cdot q$, since $0<b-1<b+1<n$, we have either ($p \mid b-1$ and $q \mid b+1$) or ($q \mid b-1$ and $p \mid b+1$), therefore, one of the factor can be found by gcd(b-1,n) and the other by n/gcd(b-1,n) or gcd(b+1,n)

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  p = gcd(b-1,n) = gcd(5371,26069) = 131, q = n/p = 199

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- The above result says that "if you can recover a pair of RSA keys, you can factoring the corresponding $n=p \cdot q$" i.e. "once a private key d is compromised, you need to choose a new pair of (n, e) instead of changing e only"

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- ✧ The above result suggests that a scheme using (n, e_1), (n, e_2), … (n, e_k) with a common n for each k participants without giving each one the value of p, q is insecure. You should not use the same n as some others even though you are not explicitly told the value of p and q.

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- The above result also suggests that if you can recover arbitrary RSA key pair, you can solve the problem of factoring n. Whenever you get an **n**, you can form an RSA system with some ***e*** (assuming gcd(e, ϕ(n))=1), then use your method to solve the private exponent ***d*** without knowing p and q, after that you can factor n.

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different things

Deterministic Encryption

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- Add randomness through padding

RSA PKCS #1 v1.5 padding

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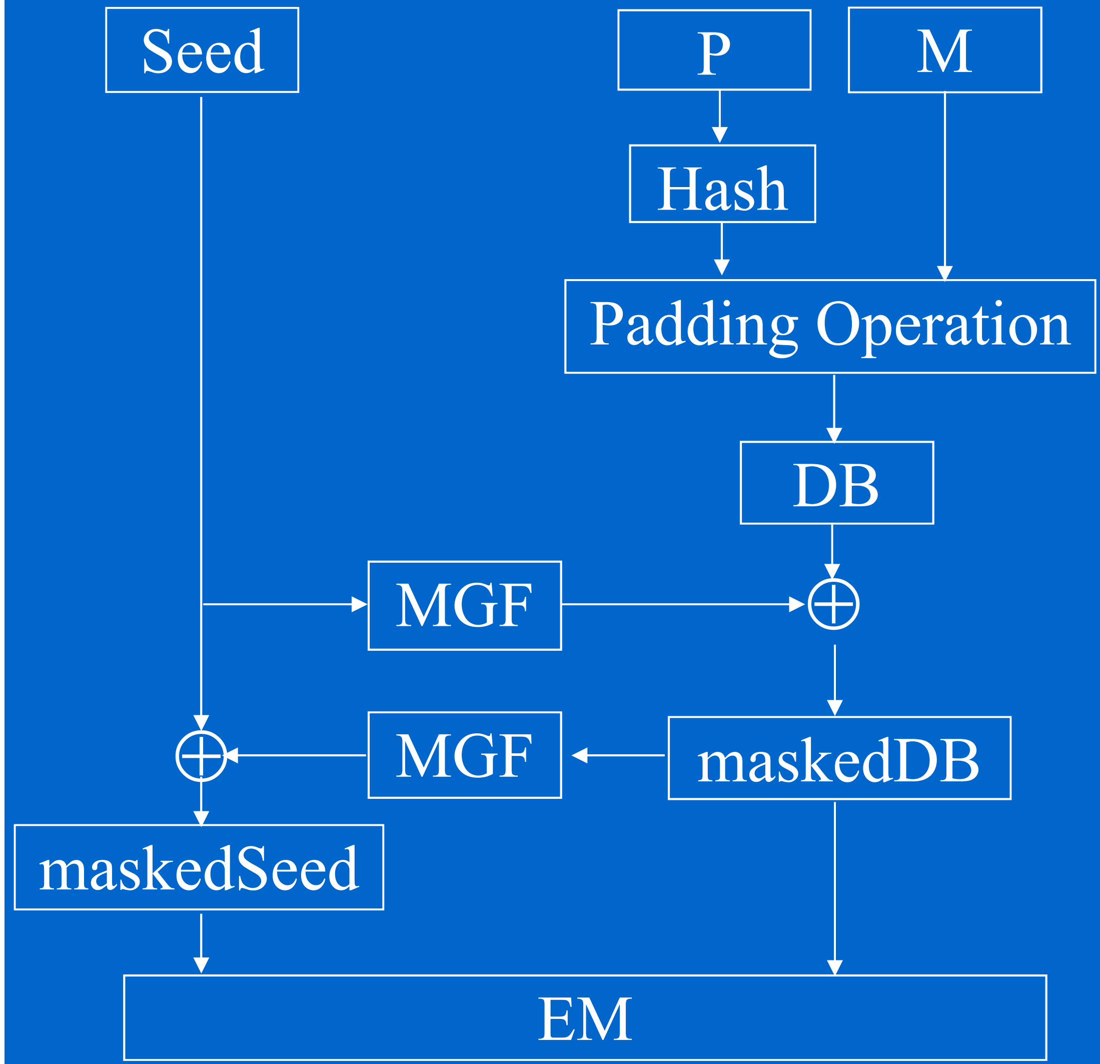
- c is now random corresponding to a fixed m, however, this only adds difficulties to the compilation of ciphertexts (a factor of 2^{64} times if PS is 8 bytes)

PKCS #1 v2 padding - OAEP



M: message (emLen-1-2hLen bytes)
P: encoding parameters,
an octet string
MGF: mask generation function
Hash: selected hash function
(hLen is the output bytes)
DB=Hash(P)||PS||01||M
PS is length emLen-||M||-2hLen-1 null bytes
Seed: hLen random bytes
dbMask: MGF(seed, emLen-hLen)
maskedDB = DB ⊕ dbMask
seedMask:
MFG(maskedDB, hLen)
maskedSeed = seed ⊕ seedMask

EM: encoded message (emLen bytes)
EM = maskedSeed||makedDB

PKCS #1 v2 padding - OAEP

- Optimal Asymmetric Encryption (OAE)
  - M. Bellare, "Optimal Asymmetric Encryption - How to Encrypt with RSA," Eurocrypt'94
- Optimal Padding in the sense that
  - RSA-OAEP is semantically secure against adaptive chosen ciphertext attackers in the random oracle model
  - the message size in a k-bit RSA block is as large as possible (make the most advantage of the bandwidth)
- Following by more efficient padding schemes:
  - $OAEP^+$, $SAEP^+$, REACT

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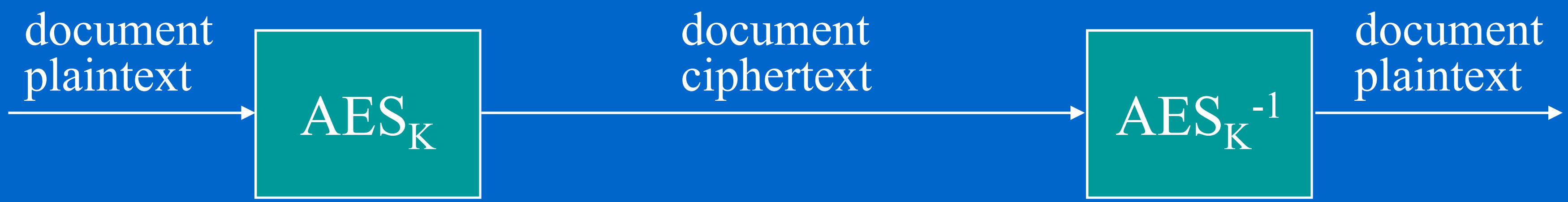

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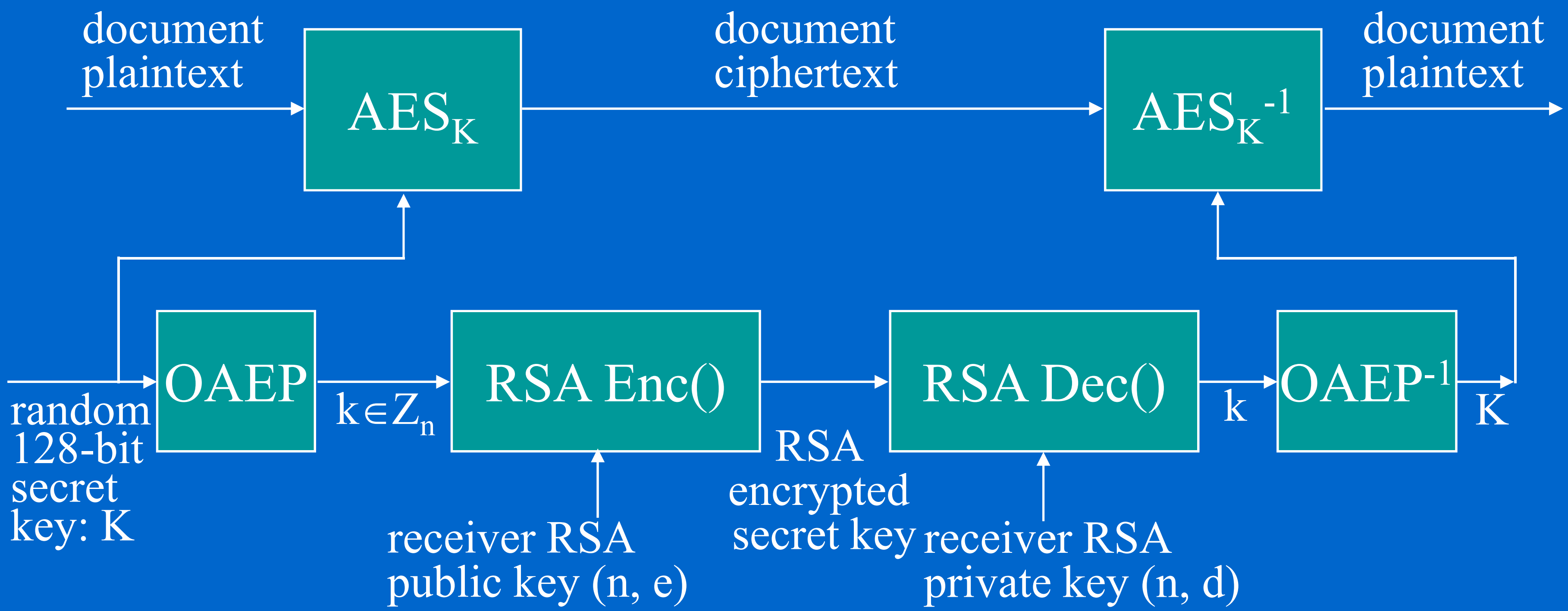

KEM/DEM

- Key/Data Encapsulation Mechnism, hybrid scheme
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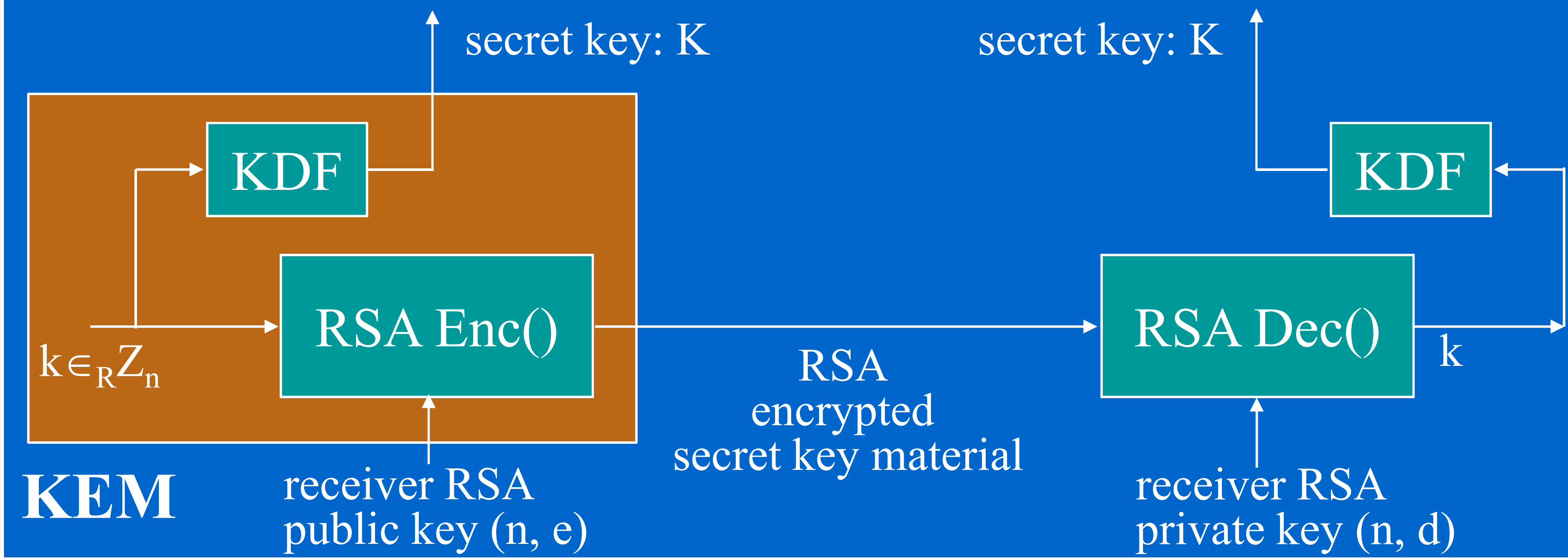

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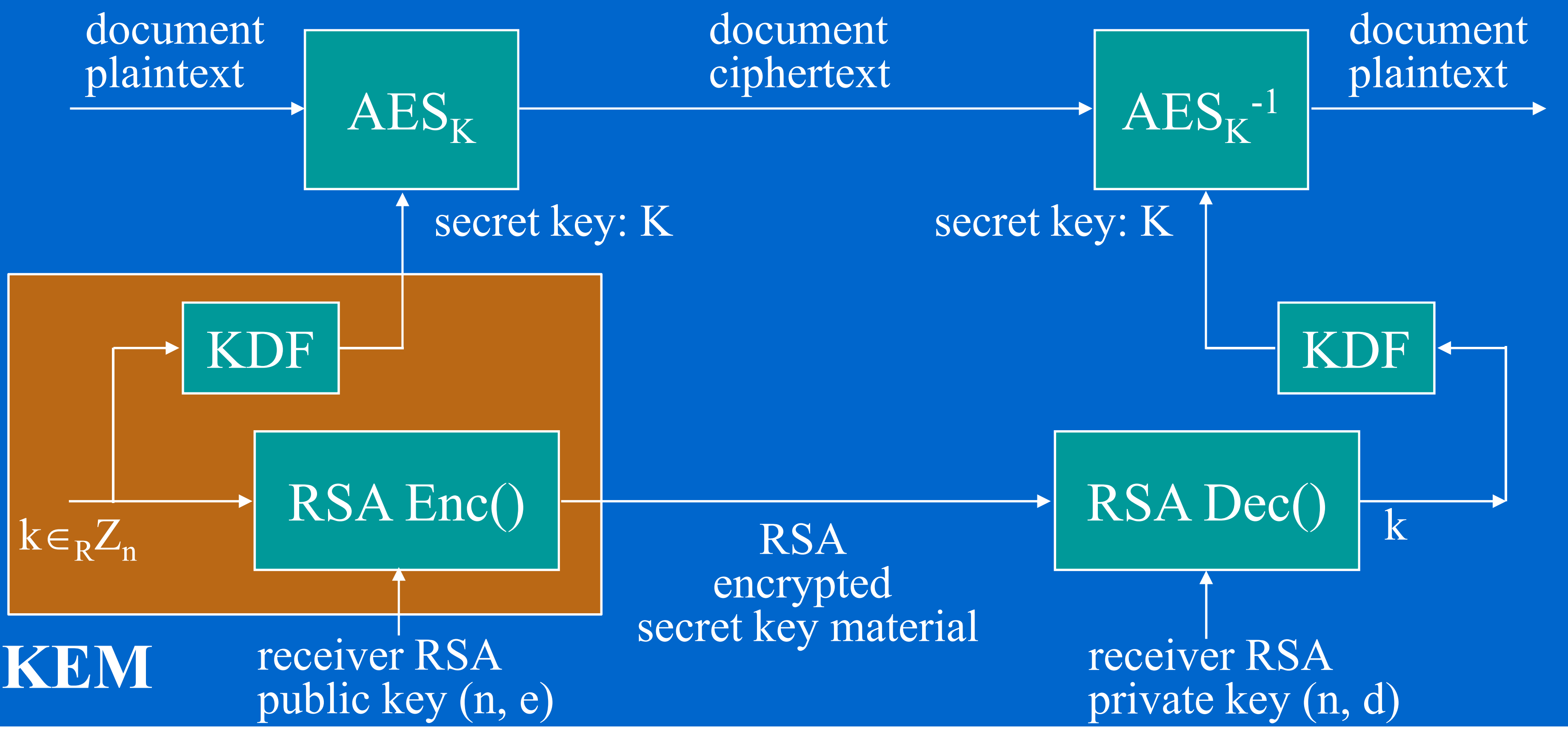

RSA Fast Decryption with CRT

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$n = p \cdot q$, p and q are large prime integers
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CRT

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4. $h \equiv (m_1 - m_2)\, qInv \pmod{p}$
5. $m = m_2 + q \cdot h$
6. if k>2, $R = r_1$, for k=3 to k do
   a. $R = R \cdot r_{i-1}$
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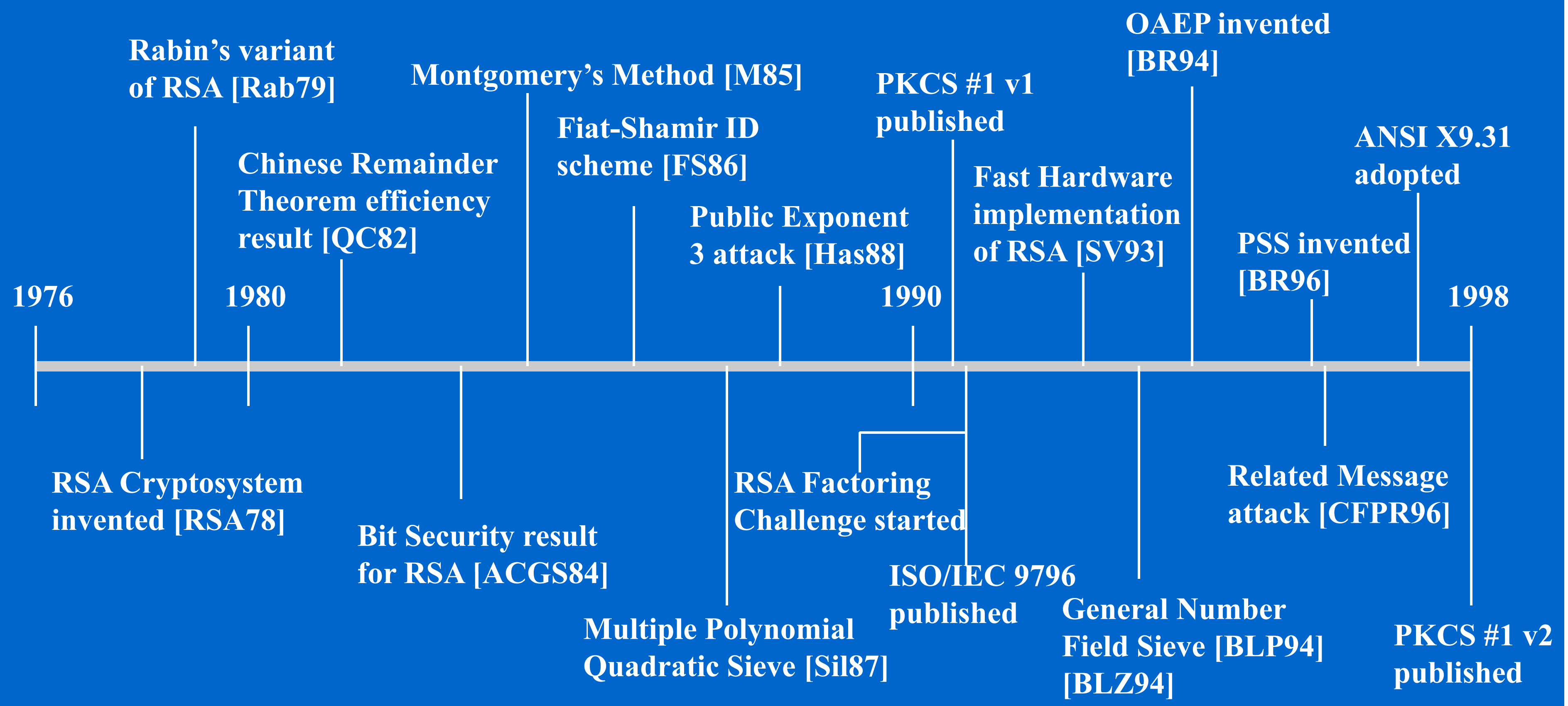
- advantages: lower computational cost for the decryption (and signature) primitives if CRT is used (also see **6.8.14**)

Factoring & RSA Timeline



Alternative PKC's

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  - very low efficiency

Miller-Rabin Primality Test

- Why does it work?

bottom line of Miller-Rabin test

- if n is prime, $a^{n-1} \equiv 1 \pmod{n}$ (Fermat Little theorem)
- therefore, if $b_k \equiv a^{2^k m} \equiv a^{n-1} \quad 1 \pmod{n}$, n must be composite
- however, there are many composite numbers that satisfy $a^{n-1} \equiv 1 \pmod{n}$, Miller-Rabin test can detect many of them
- $b_0, b_1, \ldots, b_{k-1}$ ($\equiv a^{(n-1)/2} \pmod{n}$) is a sequence s.t. $b_{i-1}^2 \equiv b_i \pmod{n}$
- we consider only $b_{k-1}^2 \equiv a^{n-1} \equiv 1 \pmod{n}$ ← n is pseudo prime
- if $b_i \equiv 1$ and $b_{i-1} \quad \pm 1$, then n is composite ← basic factoring principle
- if $b_i \equiv 1$ and $b_{i-1} \equiv 1$, consider b_{i-1} and then $b_{i-2}\ldots$
  if $b_0 \equiv 1$, could be prime, no guarantee
- if $b_i \equiv 1$ and $b_{i-1} \equiv -1$ ($b_{i-2} \quad \pm 1$), could be prime, no guarantee

there is no chance to apply basic factoring principle

Miller-Rabin Primality Test

- In summary:

  $b_0, b_1, b_2, \ldots b_{i-1}, b_i, \ldots b_k$

  there are four cases:
  - Case 1: $b_k \neq 1$ n is a composite number
  - Case 2: $b_k = 1$, let i be the minimal i, $k \geq i > 0$ such that $b_i = 1$ and $b_{i-1} \neq \pm 1$ n is a composite number (with nontrivial factors calculated)
  - Case 3: $b_k = 1$, let i be the minimal i, $k \geq i > 0$ such that $b_i = 1$ and $b_{i-1} = -1$ a pseudo prime number
  - Case 4: $b_k = 1$, $b_0 = 1$ a pseudo prime number

---

4 possible sequences for $b_0, b_1, b_2, \ldots b_{i-1}, b_i, \ldots b_k$:

342, 22, 5, 1, 1, 1, 1, …, 1	composite, factored
45, 5634, 325, 213, -1, 1, …, 1	possibly prime
1, 1, 1, …, 1	possibly prime
214, 987, …, 8931, 321, 134	composite

M-R Test: Prime Modulus

- consider n being a *prime number* p
- p-1 is an even number, therefore, let $p-1=2^k \cdot m$, m is odd
- choose one $a \in_R Z_p^*$, let r be the smallest integer s.t.
  $a^r \equiv 1 \pmod p$, i.e. r is the order of a modulo p, $ord_p(a)$
- (exercise 3.9) $a^{p-1} \equiv 1 \pmod p \Rightarrow r \mid p-1$
- because $r \mid p-1$ $(= 2^k \cdot m)$, one of $\{m, 2 \cdot m, 2^2 \cdot m, \ldots 2^k \cdot m\}$ might be r (probability reduces if m has many factors)
- Case 1: if "$2^i \cdot m$ (for some $i>0$) is r", $a^{2^{i-1} \cdot m}$ must be -1
  - r is the smallest integer s.t. $a^r \equiv 1 \Rightarrow$ square root of a^r must be -1
  - $\{a^m, a^{2 \cdot m}, \ldots a^{2^i \cdot m}\}$ is $\{?, ?, -1, 1, \ldots 1\}$
- Case 2: if "none of $2^i \cdot m$ is r" or "m is r", $a^{2^i \cdot m}$ must all be 1,
  - $\{a^m, a^{2 \cdot m}, \ldots a^{2^i \cdot m}\}$ is $\{1, 1, 1, 1, \ldots 1\}$
  - try some other $a \in Z_p^*$

Miller-Rabin Primality Test

Why does it work??? an inside view

- $b_i \equiv 1 \pmod{n}$ and b_{i-1} $\pm 1 \pmod{n}$ happens when $b_i \equiv 1 \pmod{p_i}$ for all prime factors p_i of n and

  $b_{i-1} \equiv 1 \pmod{p_i}$ for some prime factors p_i but
  $b_{i-1} \equiv -1 \pmod{q_i}$ for other prime factors q_i

Note: for a prime modulus p, $a^{ord_p(a)} \equiv 1 (\bmod\ p)$

if $ord_p(a)$ is even then $a^{ord_p(a)/2} \equiv -1 (\bmod\ p)$

- e.g. $n = 561 = 3 \times 11 \times 17$, $560 = 16 \times 35 = 2^4 \times 35$
  let $a = 2$

$b_0 \equiv 263 \pmod{561} \equiv -1 \pmod{3} \equiv -1 \pmod{11} \equiv 8 \pmod{17}$
$b_1 \equiv 166 \pmod{561} \equiv 1 \pmod{3} \equiv 1 \pmod{11} \equiv -4 \pmod{17}$
$b_2 \equiv 67 \pmod{561} \equiv 1 \pmod{3} \equiv 1 \pmod{11} \equiv -1 \pmod{17}$
$b_3 \equiv 1 \pmod{561} \equiv 1 \pmod{3} \equiv 1 \pmod{11} \equiv 1 \pmod{17}$

i.e. inconsistent progress w.r.t each prime factor

Subset Sum Problem is NP-Complete

- Subset Sum Problem (SSP)

  Given a set B of positive numbers and a number d

  - Search SSP: find a subset $\{b_j\} \subseteq B$ s.t. $d = \sum b_j$
  - Decision SSP: decide if there exists a subset $\{b_j\} \subseteq B$ s.t. $d = \sum b_j$
  - Decision SSP is equivalent to Search SSP: (by elimination)

- Subset Sum Problem is NP-complete
  - Cook-Levin Thm: Satisfiability Problem (SAT) is NP-Complete
  - SAT $\leq_M$ SSP: there exists a poly-time reduction to convert a formula ϕ to an instance <B,d> of SSP problem
    - If the formula ϕ is satisfiable, $\langle B,d \rangle \in$ SSP
    - If $\langle B,d \rangle \in$ SSP, formula ϕ is satisfiable

  Therefore, SSP is also NP-complete

SAT $\leq_M$ D-Subset Sum

- Given a formula ϕ with k clauses $C_1, C_2, \ldots, C_k$ and n variables
  - For each variable x, create 2 integers n_{xt} and n_{xf}
  - For each clause C_j of lengh ℓ_j, create ℓ_j-1 integers $m_{j1}, m_{j2}, \ldots$
  - Choose t so that T must contain exactly one of each (n_{xt} or n_{xf}) pairs and at least one from each clause
- This construction can be carried out in poly-time
- ϕ is satisfiable iff there exists solution to this SSP

SAT $\leq_M$ D-Subset Sum (cont'd)

Example: $(x \vee y \vee z) \wedge (\neg x \vee \neg a) \wedge (a \vee b \vee \neg y \vee \neg z)$

	x	y	z	a	b	C_1	C_2	C_3
n_{xt}	1	0	0	0	0	1	0	0
n_{xf}	1	0	0	0	0	0	1	0
n_{yt}		1	0	0	0	1	0	0
n_{yf}		1	0	0	0	0	0	1
n_{zt}			1	0	0	1	0	0
n_{zf}			1	0	0	0	0	1
n_{at}				1	0	0	0	1
n_{af}				1	0	0	1	0
n_{bt}					1	0	0	1
n_{bf}					1	0	0	0
m_{11}						1	0	0
m_{12}						1	0	0
m_{21}						0	1	0
m_{31}						0	0	1
m_{32}						0	0	1
m_{33}						0	0	1
t	1	1	1	1	1	3	2	4

Encode all numbers with a base larger than all entries of t e.g. 10