

RSA Cryptosystem



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Naïve Public Key System

- Encryption and decryption algorithm are not the same
- Public/private key pair: private key is related to public key but can not be easily derived from public key
- Illustrating example:

$$m \in Z_{11}^*$$

$$m * 1 = m \pmod{11}$$

$$m * \underbrace{8}_{\text{encryption}} * \underbrace{8^{-1}}_{\text{decryption}} = m \pmod{11}$$

8 is the public key
 $m * 8$ is the ciphertext
 8^{-1} is the private key (if nobody can derive this from the public key, then this system is secure)

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Knapsack (Subset Sum) PKC

- Merkel and Hellman, "Hiding Information and Signatures in Trapdoor Knapsacks," IT-24, 1978
 - a good application of an **NP problem** on designing public key cryptosystem **no longer secure**
- Super-increasing sequence:**
 $\{a_1, a_2, \dots, a_n\}$ such that $a_i > \sum_{j=0}^{i-1} a_j$ e.g. 1, 3, 5, 10, 20, 40
- Note:** 1. Given a number c , finding a subset $\{a_j\}$ s.t. $c = \sum_j a_j$ is an easy problem, e.g. $48 = 40 + 5 + 3$
- 2. Sum of every subset S , $\sum_{j \in S} a_j < 2 \cdot a_M$ where $a_M = \max_{j \in S} \{a_j\}$
- 3. Every possible subset sum is unique
 pf: given x , assume $x = \sum_{j \in S} a_j = \sum_{j \in T} a_j$, where $S \neq T$, assume $\max_{j \in S} \{a_j\} \neq \max_{j \in T} \{a_j\}$

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Knapsack (Subset Sum) PKC

- choose a number b in Z_p^* , e.g. $p = 101$, $b = 23$, and convert the **super-increasing sequence** to a **normal knapsack sequence** $B = \{b_1, b_2, \dots, b_n\}$ where $b_i \equiv a_i \cdot b \pmod{p}$
 e.g. $A = \{1, 3, 5, 10, 20, 40\} \Rightarrow B = \{23, 69, 14, 28, 56, 11\}$
- Since $\gcd(b, p) = 1$, this conversion is **invertible**, i.e.
 $a_i \equiv b_i \cdot b^{-1} \pmod{p}$
 e.g. $b^{-1} \equiv 22 \pmod{101}$ such that $b \cdot b^{-1} \equiv 1 \pmod{p}$
- Given a number d , finding a subset $\{b_j\} \subseteq B$ s.t.
 $d = \sum_j b_j \pmod{p}$
 is an NP-complete problem, e.g. $94 = 11 + 14 + 69$

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Knapsack (Subset Sum) PKC

✧ Encryption:

- * **public key:** normal knapsack seq. $B = \{23, 69, 14, 28, 56, 11\}$
- * message m , $0 \leq m < 2^6$, e.g. $(60)_{10} = (111100)_2$
- * sum up the corresponding elements of '1' bits, e.g.
 $23 + 69 + 14 + 28 = \mathbf{134}$ is the ciphertext

✧ Decryption:

- * **private key:** $b^{-1}=22$, $p=101$, $A=\{1, 3, 5, 10, 20, 40\}$
- * calculate $\mathbf{134} * 22 \bmod 101 = \mathbf{19}$
- * use the corresponding super-increasing knapsack seq. $A=\{1, 3, 5, 10, 20, 40\}$ to decrypt as follows:
 - ✧ $19 < 40$, mark a '0'
 - ✧ $19 < 20$, mark a '0'
 - ✧ $19 \geq 10$, mark a '1' and subtract 10 from 19
 - ✧ $9 \geq 5$, mark a '1' and subtract 5 from 9
 - ✧ $4 \geq 3$, mark a '1' and subtract 3 from 4
- * recovered message is $(111100)_2 = (60)_{10}$

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Knapsack (Subset Sum) PKC

✧ Why does it work?

let the plaintext be $(111100)_2$

ciphertext $c = b_1 + b_2 + b_3 + b_4$

$$\equiv a_1 b + a_2 b + a_3 b + a_4 b \pmod{p}$$

decryption: $c b^{-1} \pmod{p} \equiv a_1 + a_2 + a_3 + a_4 \pmod{p}$

is a subset sum problem of a
super-increasing sequence

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RSA and Rabin

- ✧ two important cryptosystems based on the difficulty of **integer factoring** (an NP problem) are introduced as follows:

✧ RSA's underlying problem

Solving e-th root modulo n is difficult

RSA function

$$y \equiv x^e \pmod{n}$$

✧ Rabin's underlying problem

Solving square root modulo n is difficult

Rabin function

$$y \equiv x^2 \pmod{n}$$

both functions are candidates for **trapdoor one way function**

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RSA and Rabin Function

- ✧ Solving e-th root of y modulo n is difficult!!!

$$y \equiv x^e \pmod{n}, \text{ where } \gcd(e, \phi(n)) = 1$$

Why don't we take (e^{-1}) -th power of y ?

$$\text{where } e^{-1} \cdot e \equiv 1 \pmod{\phi(n)}$$

$$\text{e.g. } n = 11 \cdot 13 = 143, e = 7$$

$$\phi(n) = 10 \cdot 12 = 120, e^{-1} = 103$$

Trouble: How do we
know $\phi(n)$?

- ✧ Solving square root of y modulo n is difficult!!!

$$y \equiv x^2 \pmod{n}$$

Why don't we take (2^{-1}) -th power of y ?

$$\text{where } 2^{-1} \cdot 2 \equiv 1 \pmod{\phi(n)}$$

$$\text{e.g. } n = 11 \cdot 13 = 143$$

$$\phi(n) = 10 \cdot 12 = 120, \gcd(2, \phi(n)) = 2$$

Remember solving square
root of y modulo a prime
number p is very easy

Trouble: $d \cdot 2 \equiv 1 \pmod{\phi(n)}$ has no solution

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RSA Public Key Cryptosystem

- ✧ R. Rivest, A. Shamir and L. Adleman, "A Method for Obtaining Digital Signatures and Public-Key Cryptosystems," Comm. ACM, pp.120-126, 1978
- ✧ Based on the *Integer Factorization* problem
- ✧ Choose two large prime numbers: p, q (keep them secret!!)
- ✧ Calculate the modulus $n = p \cdot q$ (make it public)
- ✧ Calculate $\Phi(n) = (p-1) \cdot (q-1)$ (keep it secret)
- ✧ Select a random integer such that $e < \Phi$ and $\gcd(e, \Phi) = 1$
- ✧ Calculate the unique integer d such that $e \cdot d \equiv 1 \pmod{\Phi}$
- ✧ **Public key:** (n, e) **Private key:** d

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RSA Encryption & Decryption

- ✧ Alice wants to encrypt a message m for Bob
- ✧ Alice obtains Bob's authentic public key (n, e)
- ✧ Alice represents the message as an integer m in the interval $[0, n-1]$
- ✧ Alice computes the modular exponentiation
$$c \equiv m^e \pmod{n}$$
- ✧ Alice sends the ciphertext c to Bob
- ✧ Bob decrypts c with his private key (n, d) by computing the modular exponentiation
$$\hat{m} \equiv c^d \pmod{n}$$

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RSA Encryption & Decryption

- ✧ Why does RSA work? **Is this really a problem???**

★ **Fact 1:** $e \cdot d \equiv 1 \pmod{\Phi} \Rightarrow e \cdot d = 1 + k \Phi$

★ **Fact 2:** $\forall m, \gcd(m, n) = 1, m^\Phi \equiv 1 \pmod{n}$
(by Euler's theorem)

★ **From Fact 2:** $\forall m, \gcd(m, n) = 1,$

$$c^d \equiv m^{ed} \equiv m^{1+k\Phi} \equiv m^{1+k(p-1)(q-1)} \equiv m \pmod{n}$$

note: 1. This only proves that for all m that are not multiples of p or q can be recovered after RSA encryption and decryption.

2. For those m that are multiples of p or q , the Euler's theorem simply does not hold, e.g. $p^\Phi \equiv 0 \pmod{p}$ and $p^\Phi \equiv 1 \pmod{q}$
which means that $p^\Phi \not\equiv 1 \pmod{n}$ from CRT.

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RSA Encryption & Decryption

- ✧ Why does RSA work?

★ **Fact 1:** $e \cdot d \equiv 1 \pmod{\Phi} \Rightarrow e \cdot d = 1 + k \Phi$

★ **Fact 2:** $\forall m, \gcd(m, p) = 1, m^{p-1} \equiv 1 \pmod{p}$
(by Fermat's Little theorem)

★ **From Fact 2:** $\forall m, \gcd(m, p) = 1$

note: this equation is trivially true when $m = kp$

$$m^{1+k(p-1)(q-1)} \equiv m \pmod{p}$$

★ **From Fact 2:** $\forall m, \gcd(m, q) = 1$

note: this equation is trivially true when $m = kq$

$$m^{1+k(p-1)(q-1)} \equiv m \pmod{q}$$

★ **From CRT:** $\forall m,$

$$c^d \equiv m^{ed} \equiv m^{1+k\Phi} \equiv m^{1+k(p-1)(q-1)} \equiv m \pmod{n}$$

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RSA Function is a Permutation

- ✧ RSA function is a permutation: (1-1 and onto, bijective)
 - ✧ Goal: “ $\forall x_1, x_2 \in \mathbb{Z}_n^*$ if $x_1^e \equiv x_2^e \pmod{n}$ then $x_1 = x_2$ ”
 - $\forall x \neq r \cdot p, x^{p-1} \equiv 1 \pmod{p}, \forall x \neq s \cdot q, x^{q-1} \equiv 1 \pmod{q}$
 - $\Rightarrow \forall k, \forall x \neq r \cdot p, x^{k\phi(n)} \equiv 1 \pmod{p}, \forall x \neq s \cdot q, x^{k\phi(n)} \equiv 1 \pmod{q}$
 - CRT $\Rightarrow \forall k, \forall x, x^{k\phi(n)+1} \equiv x \pmod{p}, x^{k\phi(n)+1} \equiv x \pmod{q}$
 - $\Rightarrow \forall k, \forall x, x^{k\phi(n)+1} \equiv x \pmod{n}$
 - * $\gcd(e, \phi(n)) = 1 \Rightarrow$ inverse of $e \pmod{\phi(n)}$ exists
 - \Rightarrow let d be the inverse s.t. $e \cdot d \equiv 1 \pmod{\phi(n)}$
 - * $\forall x_1, x_2 \in \mathbb{Z}_n^*$ if $x_1^e \equiv x_2^e \pmod{n}$
 - $\Rightarrow (x_1^e)^d \equiv (x_2^e)^d \pmod{n}$
 - $\Rightarrow (x_1)^{e \cdot d} \equiv (x_2)^{e \cdot d} \pmod{n}$
 - $\Rightarrow x_1 \equiv x_2 \pmod{n}$
- Note: Euler Thm is valid only when $x \in \mathbb{Z}_n^*$

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RSA Cryptosystem

- ✧ Most popular PKC in practice
- ✧ Tens of dedicated crypto-processors are specifically designed to perform modular multiplication in a very efficient way.
- ✧ **Disadvantage:** long key length, complex key generation scheme, deterministic encryption
- ✧ For acceptable level of security in commercial applications, 1024-bit (300 digits) keys are used. For a symmetric key system with comparable security, about 100 bits keys are used.
- ✧ In constrained devices such as smart cards, cellular phones and PDAs, it is hard to store, communicate keys or handle operations involving large integers

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Matlab examples

✧ rsatest.m

```
* maple('p := nextprime(1897345789)')
* maple('q := nextprime(278478934897)')
* maple('n := p*q');
* maple('x := 101');
* maple('e := nextprime(12345678)')
* maple('d := e&^(-1) mod ((p-1)*(q-1))')
* maple('y := x&^(e) mod n')
* maple('xp := y&^(d) mod n')
```

Very likely to be relatively prime with $(p-1)(q-1)$

extended Euclidean algo.

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Python gmpy2

```
from gmpy2 import mpz, next_prime, invert, powmod
```

```
p = next_prime(mpz(1897345789)) # 1897345817
q = next_prime(mpz(278478934897)) # 278478934961
n = p * q # 528370842370868408137
phi = (p-1)*(q-1) # 528370842090492127360
e = next_prime(mpz(1897345789)) # 1897345817
d = invert(e, phi) # 139387972146660337833

plaintext = 101
ciphertext = powmod(plaintext, e, n) # 479679342785929350234
decrypted = powmod(ciphertext, d, n) # 101
```

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Rabin Cryptosystem (1/3)

- ✧ M.O. Rabin, “Digitalized Signatures and Public-key Functions As Intractable As Factorization”, Tech. Rep. LCS/TR212, MIT, 1979
- ✧ Choose two large prime numbers: p, q (keep them secret!!)
- ✧ Calculate the modulus $n = p \cdot q$ (make it public)
- ✧ **Public Key** n
- ✧ **Private Key** p, q

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Rabin Cryptosystem (2/3)

- ✧ Alice want to encrypt a message m (with some fixed format) for Bob
- ✧ Alice obtains Bob's authentic public key n
- ✧ Alice represents the message as an integer m in the interval $[0, n - 1]$
- ✧ Alice computes the modular square
$$c \equiv m^2 \pmod{n}$$
- ✧ Alice sends the ciphertext c to Bob
- ✧ Bob decrypts c using his private key p and q
- ✧ Bob computes the four square roots $\pm m_1, \pm m_2$ using CRT, one of them satisfying the fixed message format is the recovered message

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Rabin Cryptosystem (3/3)

- ✧ The range of the Rabin function is not the whole set of Z_n^* (compare with RSA).
 - ★ The range covers all the quadratic residues. (for a prime modulus, the number of quadratic residues in Z_p^* is $(p-1)/2$; for a composite integer $n=p \cdot q$, the number of quadratic residues in Z_n^* is $(p-1)(q-1)/4$)
 - ★ In order to let the Rabin function have inverse, it is necessary to make the Rabin function a permutation, ie. 1-1 and onto. Therefore, the number of elements in the domain of the Rabin function should also be $(p-1)(q-1)/4$ for $n=p \cdot q$. There are 4 possible numbers with their square equal to y , and we have to make 3 of them illegal.

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Number of Quadratic Residues

- ✧ For a prime modulus p : number of QR_p 's in Z_p^* is $(p-1)/2$
 pf: find a primitive g , at least $\{g^2, g^4, \dots, g^{p-1}\}$ are QR_p 's
 assume there are $(p+1)/2$ QRs,
 since there are exactly two square roots of a QR modulo p
 there are $p+1$ square roots for these $(p+1)/2$ QRs, i.e. there must be at least two pairs of square roots are the same (pigeon-hole), i.e. two out of these $(p+1)/2$ QRs are the same, contradiction
- ✧ For a composite modulus $p \cdot q$: number of QR_n 's in $Z_{p \cdot q}^*$ is $(p-1)(q-1)/4$
 pf: find a common primitive in Z_p^* and Z_q^* g , at least $\{g^2, g^4, \dots, g^{p-1}, \dots, g^{q-1}, \dots, g^{\lambda(n)}\}$ are QR_n 's, where $\lambda(n) = \text{lcm}(p-1, q-1)$ can be as large as $(p-1)(q-1)/2$, this set has $(p-1)(q-1)/4$ distinct elements
 assume there are $(p-1)(q-1)/4 + 1$ QR_n 's in Z_n^* , since there are four square roots of a QR modulo $p \cdot q$, these QR_n 's have $(p-1)(q-1) + 4$ square roots in total. There must be some repeated elements in this QR_n , therefore, there are at most $(p-1)(q-1)/4$ QR_n 's in Z_n^*

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Matlab examples

```

❖ maple('p:= nextprime(189734535789)') % 189734535811 = 4 k + 3
❖ maple('p mod 4')
❖ maple('q:= nextprime(27847815934897)') % 27847815934931 = 4 k + 3
❖ maple('q mod 4')
❖ maple('n:=p*q');
❖ maple('x:=070411111422141711030000') % text2int('helloworld')
❖ maple('c:= x&^2 mod n')

❖ maple('c1:= c mod p')
❖ maple('r1:= c1&^((p+1)/4) mod p') % maple('r1&^2 mod p')

❖ maple('c2:= c mod q')
❖ maple('r2:= c2&^((q+1)/4) mod q') % maple('r2&^2 mod q')

❖ maple('m1:= chrem([r1, r2], [p, q])') % 3704440302544264662351219
❖ maple('m2:= chrem([-r1, r2], [p, q])') % 70411111422141711030000
❖ maple('m3:= chrem([r1, -r2], [p, q])') % 5213281318342160554284041
❖ maple('m4:= chrem([-r1, -r2], [p, q])') % 1579252127220037602962822

```

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Security of the RSA Function

- ❖ **Break RSA** means ‘inverting RSA function without knowing the trapdoor’
- ❖ Factor the modulus \Rightarrow Break RSA
 - * If we can factor the modulus, we can break RSA
 - * If we can break RSA, we don’t know whether we can factor the modulus...**open problem** (with negative evidences)
- ❖ Factor the modulus \Leftrightarrow Calculate private key d
 - * If we can factor the modulus, we can calculate the private exponent d (the trapdoor information).
 - * If we have the private exponent d, we can factor the modulus.

$$y \equiv x^c \pmod{n}$$

will be illustrated later after factorization 22

Security of Rabin Function

- ❖ Security of Rabin function *is equivalent to* integer factoring
- ❖ inverting ‘ $y \equiv f(x) \equiv x^2 \pmod{n}$ ’ without knowing p and q \Leftrightarrow factoring n

- * \Leftarrow
 - if you can factor $n = p \cdot q$ in polynomial time
 - you can solve $y \equiv x_1^2 \pmod{p}$ and $y \equiv x_2^2 \pmod{q}$ easily
 - using CRT you can find x which is $f^{-1}(y)$

- * \Rightarrow
 - given a quadratic residue y if you can find the four square roots $\pm x_1$ and $\pm x_2$ for y in polynomial time
 - you can factor n by trying $\gcd(x_1 - x_2, n)$ and $\gcd(x_1 + x_2, n)$

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Basic Factoring Principle (1/4)

- ❖ Let n be an integer and suppose there exist integers x and y with $x^2 \equiv y^2 \pmod{n}$, but $x \not\equiv \pm y \pmod{n}$. Then ❶ n is composite, ❷ both $\gcd(x-y, n)$ and $\gcd(x+y, n)$ are nontrivial factors of n.

Proof:

let $d = \gcd(x-y, n)$.

Case 1: assume $d = n \Rightarrow x \equiv y \pmod{n}$ contradiction

Case 2: assume d is 1 (the trivial factor)

$$x^2 \equiv y^2 \pmod{n} \Rightarrow x^2 - y^2 = (x-y)(x+y) = k \cdot n$$

$$d=1 \text{ means } \gcd(x-y, n)=1 \Rightarrow$$

$$n \mid x+y \Rightarrow x \equiv -y \pmod{n} \text{ contradiction}$$

Case 1 and 2 implies that $1 < d < n$

i.e. d must be a nontrivial factor of n

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Basic Factoring Principle (2/4)

- ✧ $x^2 \equiv y^2 \pmod{p}$ implies $x \equiv \pm y \pmod{p}$ since $p \mid (x+y)(x-y)$ implies $p \mid (x+y)$ or $p \mid (x-y)$,
i.e. $x \equiv -y \pmod{p}$ or $x \equiv y \pmod{p}$
- ✧ $x^2 \equiv y^2 \pmod{n}$
 $pq \mid (x+y)(x-y)$ implies the following 4 possibilities
 1. $pq \mid (x+y)$ i.e. $x \equiv -y \pmod{n}$
 2. $pq \mid (x-y)$ i.e. $x \equiv y \pmod{n}$
 3. $p \mid (x+y)$ and $q \mid (x-y)$ i.e. $x \equiv -y \pmod{p}$ and $x \equiv y \pmod{q}$
 4. $q \mid (x+y)$ and $p \mid (x-y)$ i.e. $x \equiv -y \pmod{q}$ and $x \equiv y \pmod{p}$
 - ★ Case 1 and case 2 are useless for factorization
 - ★ Case 3 leads to the factorization of n , i.e. $\gcd(x+y, n) = p$ and $\gcd(x-y, n) = q$
 - ★ Case 4 leads to the factorization of n , i.e. $\gcd(x+y, n) = q$ and $\gcd(x-y, n) = p$

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Basic Factoring Principle (3/4)

- ✧ This principle is used in *almost all factoring algorithms*.
- ✧ Why is it working?
 - ★ take $n = p \cdot q$ (p and q are prime) for example
 - ★ $x^2 \equiv y^2 \pmod{n}$ implies $x^2 \equiv y^2 \pmod{p}$ and $x^2 \equiv y^2 \pmod{q}$
 - ★ we know ' $x \equiv \pm y \pmod{p}$ ' are the only solution to $x^2 \equiv y^2 \pmod{p}$ ' and ' $x \equiv \pm y \pmod{q}$ ' are the only solution to $x^2 \equiv y^2 \pmod{q}$ '
 - ★ therefore, from CRT we know $x^2 \equiv y^2 \pmod{n}$ has four solutions,

$x \equiv y \pmod{p}$ and $x \equiv y \pmod{q}$	\Rightarrow	$x \equiv y \pmod{n}$
$x \equiv -y \pmod{p}$ and $x \equiv -y \pmod{q}$	\Rightarrow	$x \equiv -y \pmod{n}$
$x \equiv y \pmod{p}$ and $x \equiv -y \pmod{q}$	\Rightarrow	$x \equiv z \pmod{n}$
$x \equiv -y \pmod{p}$ and $x \equiv y \pmod{q}$	\Rightarrow	$x \equiv -z \pmod{n}$
 - ★ as long as we have z (where $z \not\equiv \pm y$), we can factor n into $\gcd(y-z, n)$ and $\gcd(y+z, n)$

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Basic Factoring Principle (4/4)

- ✧ Ex: Consider the roots of $4 \pmod{35}$, i.e. solving x from $x^2 \equiv 4 \pmod{35}$
 - ★ try to take square root of both sides,
we find $x = \pm 2$ or ± 12
 - ★ i.e. $12^2 \equiv 2^2 \pmod{35}$, but $12 \not\equiv \pm 2 \pmod{35}$
 - ★ therefore 35 is composite
 - ★ $\gcd(12-2, 35) = 5$ is a nontrivial factor of 35
 - ★ $\gcd(12+2, 35) = 7$ is a nontrivial factor of 35

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Miller-Rabin Test

Is n a composite number?

- ✧ Let $n > 1$ be odd, write $n-1 = 2^k \cdot m$ with m being odd
- ✧ Choose a random integer a with $1 < a < n-1$
- ✧ Compute $b_0 \equiv a^m \pmod{n}$
if $b_0 \equiv \pm 1 \pmod{n}$, stop, n is probably prime
- ✧ Compute $b_1 \equiv b_0^2 \pmod{n}$
if $b_1 \equiv 1 \pmod{n}$, stop, $\gcd(b_0-1, n)$ is a factor of n
if $b_1 \equiv -1 \pmod{n}$, stop, n is probably prime
- ✧ Compute $b_2 \equiv b_1^2 \pmod{n}$
.....
- ✧ Compute $b_{k-1} \equiv b_{k-2}^2 \pmod{n}$
if $b_{k-1} \equiv 1 \pmod{n}$, stop, $\gcd(b_{k-2}-1, n)$ is a factor of n
if $b_{k-1} \equiv -1 \pmod{n}$, stop, n is probably prime
- ✧ Compute $b_k \equiv b_{k-1}^2 \pmod{n}$
if $b_k \equiv 1 \pmod{n}$, stop, $\gcd(b_{k-1}-1, n)$ is a factor of n
otherwise n is composite (Fermat Little Thm, $b_k \equiv a^{n-1} \pmod{n}$)

n will pass Fermat test
 n is called pseudo prime
with respect to base a

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Miller-Rabin Test Illustrated

$$n-1 = 2^k \cdot m$$

$$b_0 \equiv a^m \pmod{n}$$

$$b_1 \equiv a^{2 \cdot m} \pmod{n}$$

...

$$b_k \equiv a^{2^k \cdot m} \equiv a^{n-1} \pmod{n}$$

Consider 4 possible cases:

① $b_0 \equiv \pm 1 \pmod{n}$

all $b_i \equiv 1 \pmod{n}$, $i=1,2,\dots,k$

there is no chance to use

Basic Factoring Principle, **abort**

② ① is not true,

$b_{i-1} \not\equiv \pm 1 \pmod{n}$ and

$b_i \equiv 1 \pmod{n}$, $i=1,2,\dots,k$

Basic Factoring Principle applied, **composite**

③ ① and ② are not true,

$b_i \equiv -1 \pmod{n}$, $i=1,2,\dots,k$

all subsequent $b_j \equiv 1 \pmod{n}$,

there is no chance to use

Basic Factoring Principle, **abort**

④ ①, ②, and ③ are not true,

$b_k \equiv a^{n-1} \pmod{n}$

if $b_k \not\equiv 1 \pmod{n}$ n is **composite**

since if n is prime, $b_k \equiv 1 \pmod{n}$

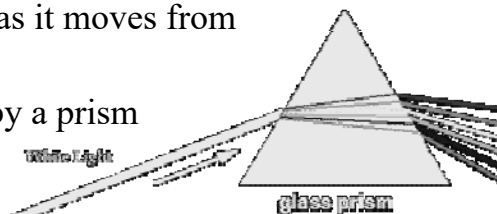
($b_k \equiv 1 \pmod{n}$ is covered by ②)

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Uncoordinated Behaviors

- Speed of light changes as it moves from one medium to another,

e.g., refraction caused by a prism



- 趣味競賽: 兩人三腳, 同心協力, ...
- Squaring a number modulo a composite number (product of different prime numbers)

	2^2	2^3	2^4	2^5	2^6	2^7	2^8
mod 11	4	8	5	10	9	7	3
mod 13	4	8	3	6	12	11	9

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When/How does Basic Factoring Principle work in M-R test?

When:

- explicitly: $b_{i-1} \not\equiv \pm 1 \pmod{n}$ and $b_i \equiv b_{i-1}^2 \equiv 1 \pmod{n}$

If n is not prime, sometimes $b_k \equiv a^{n-1} \pmod{n}$ but often $b_k \equiv a^{r\phi(n)} \pmod{n}$ as in universal exponent factoring

How:

- implicitly: let $p \mid n$ and $q \mid n$ (p, q be two factors of n)
 $b_{i-1}^2 \equiv 1 \pmod{p}$ and $b_{i-1}^2 \equiv 1 \pmod{q}$
 but either $b_{i-1} \not\equiv 1 \pmod{p}$ or $b_{i-1} \not\equiv 1 \pmod{q}$
- catching the moment that b_0, b_1, \dots behave differently while taking square in $(\text{mod } p)$ component and $(\text{mod } q)$ component

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Miller-Rabin Test Example

e.g. $n = 561$

A Carmichael number: pass the Fermat test for all bases

$$n-1 = 560 = 16 \cdot 35 = 2^4 \cdot 35$$

let $a = 2$

$$b_0 \equiv 2^{35} \equiv 263 \pmod{561}$$

$$b_1 \equiv b_0^2 \equiv 2^{2 \cdot 35} \equiv 166 \pmod{561}$$

$$b_2 \equiv b_1^2 \equiv 2^{2^2 \cdot 35} \equiv 67 \pmod{561}$$

$$b_3 \equiv b_2^2 \equiv 2^{2^3 \cdot 35} \equiv 1 \pmod{561}$$

561 is composite ($3 \cdot 11 \cdot 17$),

$\gcd(b_2-1, 561) = 33$ is a factor

Note: $3-1=2$, $11-1=2 \cdot 5$, $17-1=2^4$

$$\phi(561) = 561(1-1/3)(1-1/11)(1-1/17) = 2 \cdot 10 \cdot 16$$

$$\gcd(\phi(561), n-1) = 80, \text{ ord}_{561}(2) \mid 80 \text{ in this case}$$

mod	3	11	17
	2	10	8
	1	1	13
	1	1	16
	1	1	1

$$\text{ord}_{17}(2) = 2^3$$

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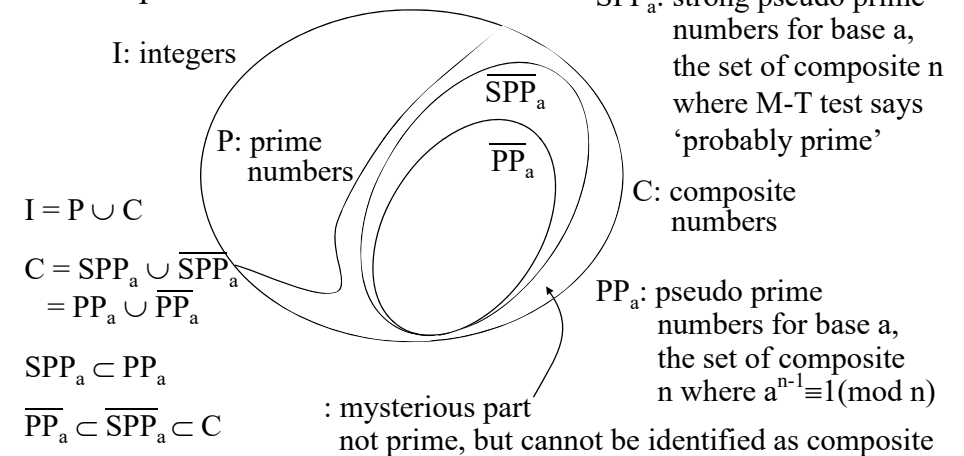
Pseudo Prime and Strong Pseudo Prime

- ✧ If n is not a prime but satisfies $a^{n-1} \equiv 1 \pmod{n}$ we say that n is a pseudo prime number for base a .
 * e.g. $2^{560} \equiv 1 \pmod{561}$
- ✧ If n is not a prime but passes the Miller-Rabin test with base a (without being identified as a composite), we say that n is a strong pseudo prime number for base a .
- ✧ Up to 10^{10} , there are 455052511 primes, there are 14884 pseudo prime numbers for the base 2, and 3291 strong pseudo prime numbers for the base 2

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Fermat and Miller-Rabin Test

- ✧ Both of these two tests are for identifying subsets of composite numbers



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Composite Witness

- ✧ Note that the **M-R test** and probably together with the **Lucas test** leave the strong pseudo prime number *an extremely small set*.
- ✧ In other words, these tests are very close to a *real 'primality test'* separating prime numbers and composite numbers.
- ✧ If you have an RSA modulus $n=p \cdot q$, you certainly can test it and find out that it is actually a composite number.
- ✧ However, these tests do not necessarily give you the factors of n in order to tell you that n is a composite number. The factors of n , i.e. p or q , are certainly a kind of witness about the fact that n is composite.
- ✧ However, there are other kind of witness that n is composite, e.g., " $2^{n-1} \pmod{n}$ does not equal to 1" is also a witness that n is composite.
- ✧ A composite number will be factored out by the M-R test only if it is a pseudo prime but it is not a strong pseudo prime number.

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Matlab Example

- ✧ `primetest(n)`
 - * Miller-Rabin test for 30 randomly chosen base a
 - * output 0 if n is composite
 - * output 1 if n is prime
 - * Matlab program can not be used for large n
 - * use `Maple isprime(n)`, one strong pseudo-primality test and one Lucas test
- ✧ `primetest(2563)`
`ans = 0`
- ✧ `factor(2563)`
`ans = 11 233`

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Questions

- ✧ What is the probability that Miller-Rabin test fails???
- ★ If n is a prime number, it will not be recognized as a composite number
- ★ If $n = p \cdot q$, but
 - $b_k \equiv a^{n-1} \equiv 1 \pmod{n}$ meets Fermat test (pseudo prime number)
 - $0 < i \leq k$ $b_i \equiv 1 \pmod{n}$ and $b_{i-1} \equiv -1 \pmod{n}$
 - meets Miller-Rabin test (strong pseudo prime number)
- or $\left[\begin{array}{l} b_i \equiv 1 \pmod{n} \equiv 1 \pmod{p} \equiv 1 \pmod{q} \\ b_{i-1} \equiv -1 \pmod{n} \equiv -1 \pmod{p} \equiv -1 \pmod{q} \end{array} \right]$
- ★ Note: $a^{pq-1} \equiv 1 \pmod{n}$
- $a^{(p-1)(q-1)} \equiv 1 \pmod{n}$
- $a^{\text{lcm}(p-1, q-1)} \equiv 1 \pmod{n}$

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Note on Primality Testing

- ✧ Primality testing is *different* from factoring
 - ★ Kind of interesting that we can tell something is composite without being able to actually factor it
- ✧ Recent result (2002) from IIT trio (Agrawal, Kayal, and Saxena)
 - ★ Recently it was shown that deterministic primality testing could be done in polynomial time
 - ✧ Complexity was like $O(n^{12})$, though it's been slightly reduced since then
 - ★ Does this mean that RSA was broken?
- ✧ Randomized algorithms like Rabin-Miller are far more efficient than the IIT algorithm, so we'll keep using those

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Finding a Random Prime

- ✧ Find a prime of around 100 digits for cryptographic usage
- ✧ Prime number theorem ($\pi(x) \approx x/\ln(x)$) asserts that the density of primes around x is approximately $1/\ln(x)$
- ✧ $x = 10^{100}$, $1/\ln(10^{100}) = 1/230$
- if we skip even numbers, the density is about $1/115$
- ✧ pick a random starting point, throw out multiples of 2, 3, 5, 7, and use Miller-Rabin test to eliminate most of the composites.
- ✧ `maple('a:=nextprime(189734535789)')`

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Factoring

- ✧ General number field sieve (GNFS): fastest

$$e^{(1.923+O(1))(\ln(n))^{1/3} (\ln(\ln(n)))^{2/3}}$$
- ✧ Quadratic sieve (QS)
- ✧ Elliptic curve method (ECM), Lenstra (1985)
- ✧ Pollard's Monte Carlo algorithm
- ✧ Continued fraction algorithm
- ✧ Trial division, Fermat factorization
- ✧ Pollard's $p-1$ factoring (1974), Williams's $p+1$ factoring (1982)
- ✧ Universal exponent factorization, exponent factorization

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Simple Factoring Methods

- ✧ Trial division:
 - * dividing an integer n by all primes $p \leq \sqrt{n}$... too slow
- ✧ Fermat factorization:
 - * e.g. $n = 295927$ calculate $n+1^2, n+2^2, n+3^2 \dots$ until finding a square, i.e. $x^2 = n + y^2$, therefore, $n = (x+y)(x-y)$... if $n = p \cdot q$, it takes on average $|p-q|/2$ steps ... too slow

assume $p > q$, $n + y^2 = p \cdot q + ((p-q)/2)^2 = (p^2 + 2pq + q^2)/4 = ((p+q)/2)^2$
 - * in RSA or Rabin, avoid p, q with the same bit length
- ✧ By-product of Miller-Rabin primality test:
 - * if n is a pseudoprime and not a strong pseudoprime, Miller-Rabin test can factor it. about 10^{-6} chance

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Universal Exponent Factorization

- * if we have an exponent r , s.t. $a^r \equiv 1 \pmod{n}$ for all a $\gcd(a, n) = 1$
- * write $r = 2^k \cdot m$ with m odd
- * choose a random a , $1 < a < n-1$
- * if $\gcd(a, n) \neq 1$, we have a factor
- * else
 - * let $b_0 \equiv a^m \pmod{n}$, if $b_0 \equiv \pm 1$ stop, choose another a
 - * compute $b_{u+1} \equiv b_u^2 \pmod{n}$ for $0 \leq u \leq k-1$,
 - * if $b_{u+1} \equiv -1$, stop, choose another a
 - * if $b_{u+1} \equiv 1$ then $\gcd(b_u - 1, n)$ is a factor (basic factoring principle)
- * Question: How do we find a universal exponent r ??? Hard
- * Note: if know $\phi(n)$, then any $r = k \phi(n)$ will do, however, knowing factors of n is a prerequisite of know $\phi(n)$
- * Note: For RSA, if the private exponent d is recovered, then $\phi(n) \mid d \cdot e - 1$, $d \cdot e - 1$ is a universal exponent

r must be even since we can take $a = -1$ $(-1)^r \equiv 1 \pmod{n}$ requires r being even

$a \equiv \pm 1$ do not work

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Universal Exponent Factorization

- ✧ E.g.

$n = 211463707796206571$; $e = 9007$; $d = 116402471153538991$
 $r = e \cdot d - 1 = 1048437057679925691936$; $\text{powermod}(2, r, n) = 1$
 let $r = 2^5 \cdot r_1$; $r_1 = 32763658052497677873$
 $\text{powermod}(2, r_1, n) = 187568564780117371 \neq \pm 1$
 $\text{powermod}(2, 2 \cdot r_1, n) = 113493629663725812 \neq \pm 1$
 $\text{powermod}(2, 4 \cdot r_1, n) = 1 \Rightarrow \gcd(2 \cdot r_1 - 1, n) = 885320963$ is a factor
- ✧ Note: $n = 211463707796206571 = 238855417 \cdot 885320963$

$238855417 - 1 = 2^3 \cdot 3 \cdot 73 \cdot 136333 = 2^{k_1} \cdot p_1$
 $885320963 - 1 = 2 \cdot 2069 \cdot 213949 = 2^{k_2} \cdot q_1$
 This method works only when k_1 does not equal k_2 .
- ✧ Exponent factorization even if r is valid for one a , you can still try the above procedure

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$p-1$ factoring (1/2)

- ✧ If one of the prime factors of n has a special property, it is sometimes easier to factor n .
 - * e.g. if $p-1$ has only small prime factors
 - * Pollard 1974
 - ✧ Algorithm
 - * Choose an integer $a > 1$ (often $a = 2$ is used)
 - * Choose a bound B
 - * Compute $b \equiv a^{B!} \pmod{n}$ as follows:
 - * $b_1 \equiv a \pmod{n}$ and $b_j \equiv b_{j-1}^j \pmod{n}$ then $b \equiv b_B \pmod{n}$
 - * Let $d = \gcd(b-1, n)$, if $1 < d < n$, we have found a factor of n

If B is larger than all the prime factors of $p-1$ (very likely) $\Rightarrow p-1 \mid B!$ therefore $b \equiv a^{B!} \equiv (a^{p-1})^k \equiv 1 \pmod{p}$, i.e. $p \mid b-1$ Fermat Little's Thm
- If $n = p \cdot q$, $p-1$ and $q-1$ both have small factors that are less than B , then $\gcd(b-1, n) = n$, (useless) however, $b \equiv a^{B!} \equiv 1 \pmod{n}$ and we can use the Universal exponent method 44

p-1 factoring (2/2)

- ✧ How do we choose B?
 - * small B will be faster but fails often
 - * large B will be very slow
- ✧ In RSA, Rabin, Paillier, or other systems based on integer factoring, usually $n=p \cdot q$, we should ensure that p-1 has at least one large prime factor.
 - * How do we do this?
 - e.g. we want to choose p around 100 digits
 - > choose a prime number p_0 around 40 digits
 - > look at integer $k \cdot p_0 + 1$ with k around 60 digits and do primality test
- ✧ Generalization:
 - Elliptic curve factorization method, Lenstra, 1985
- ✧ Best records: p-1: 34 digits (113 bits), ECM: 47 digits (143 bits)

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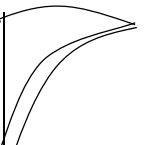
Quadratic Sieve (1/4)

- ✧ Example: factor $n = 3837523$
 - * form the following relations
 - $9398^2 \equiv 5^5 \cdot 19 \pmod{3837523}$
 - $19095^2 \equiv 2^2 \cdot 5 \cdot 11 \cdot 13 \cdot 19 \pmod{3837523}$
 - $1964^2 \equiv 3^2 \cdot 13^3 \pmod{3837523}$
 - $17078^2 \equiv 2^6 \cdot 3^2 \cdot 11 \pmod{3837523}$
 - * multiply the above relations
 - $(9398 \cdot 19095 \cdot 1964 \cdot 17078)^2 \equiv (2^4 \cdot 3^2 \cdot 5^3 \cdot 11 \cdot 13^2 \cdot 19)^2$
 - $2230387^2 \equiv 2586705^2$
 - * since $2230387 \not\equiv \pm 2586705 \pmod{3837523}$
 - * $\gcd(2230387 - 2586705, 3837523) = 1093$ is one factor of n
 - * the other factor is $3837523 / 1093 = 3511$

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Quadratic Sieve (2/4)

- ✧ Quadratic? $x^2 \equiv \text{product of small primes}$
- ✧ How do we construct these useful relations systematically?
- ✧ Properties of these relations:
 - * product of small primes called factor base
 - * make all prime factors appear even times
- ✧ Put these relations in a matrix

	2	3	5	7	11	13	17	19	
9398	0	0	5	0	0	0	0	1	
19095	2	0	1	0	1	1	0	1	
1964	0	2	0	0	0	3	0	0	
17078	6	2	0	0	1	0	0	0	
8077	1	0	0	0	0	0	0	1	
3397	5	0	1	0	0	2	0	0	
14262	0	0	2	2	0	1	0	0	

Pick rows where sums of each column are even

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Quadratic Sieve (3/4)

- ✧ Look for linear dependencies mod 2 among the rows
 - * $1\text{st} + 5\text{th} + 6\text{th} = (6, 0, 6, 0, 0, 2, 0, 2) \equiv \mathbf{0} \pmod{2}$
 - * $1\text{st} + 2\text{nd} + 3\text{rd} + 4\text{th} = (8, 4, 6, 0, 2, 4, 0, 2) \equiv \mathbf{0} \pmod{2}$
 - * $3\text{rd} + 7\text{th} = (0, 2, 2, 2, 0, 4, 0, 0) \equiv \mathbf{0} \pmod{2}$
- ✧ When we have such a dependency, the product of the numbers yields a square.
 - * $(9398 \cdot 8077 \cdot 3397)^2 \equiv 2^6 \cdot 5^6 \cdot 13^2 \cdot 19^2 \equiv (2^3 \cdot 5^3 \cdot 13 \cdot 19)^2$
 - * $(9398 \cdot 19095 \cdot 1964 \cdot 17078)^2 \equiv (2^3 \cdot 3^2 \cdot 5^3 \cdot 11 \cdot 13^2 \cdot 19)^2$
 - * $(1964 \cdot 14262)^2 \equiv (3 \cdot 5 \cdot 7 \cdot 13^2)^2$
- ✧ Looking for those $x^2 \equiv y^2$ but $x \neq \pm y$

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Quadratic Sieve (4/4)

✧ How do we find numbers x s.t.

$$x^2 \equiv \text{product of small primes?}$$

* produce squares that are slightly larger than a multiple of n

e.g. $\lfloor \sqrt{i \cdot n} + j \rfloor$ for small j

the square is approximately $i \cdot n + 2j\sqrt{i \cdot n} + j^2$

which is approximately $2j\sqrt{i \cdot n} + j^2 \pmod{n}$

$$8077 = \lfloor \sqrt{17n} + 1 \rfloor$$

$$9398 = \lfloor \sqrt{23n} + 4 \rfloor$$

Probably because this number is small, the factors of it should not be too large. However, there are a lot of exceptions. So it takes time. Also, there are a lot of other methods to generate qualified x values.

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The RSA Challenge

✧ 1977 Rivest, Shamir, Adleman US\$100

* given RSA modulus n , public exponent e , ciphertext c

$n = 114381625757888867669235779976146612010218296721242362$
 $562561842935706935245733897830597123563958705058989075$
 147599290026879543541

$e = 9007$

$c = 968696137546220614771409222543558829057599911245743198$
 $746951209308162982251457083569314766228839896280133919$
 90551829945157815154

* Find the plaintext message

✧ 1994 Atkins, Lenstra, and Leyland

* use 524339 small primes (less than 16333610)

* plus up to two large primes ($16333610 \sim 2^{30}$)

* 1600 computers, 600 people, 7 months

* found 569466 ' $x^2 \equiv \text{small products}$ ' equations, out of which only 205 linear dependencies were found

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Factorization Records

Year	Number of digits
1964	20
1974	45
1984	71
1994	129 (429 bits)
1999	155 (515 bits)
2003	174 (576 bits)

Next challenge
RSA-640

31074182404900437213507500358885679300373460228427
 27545720161948823206440518081504556346829671723286
 78243791627283803341547107310850191954852900733772
 4822783525742386454014691736602477652346609

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Security of the RSA Function

✧ **Break RSA** means 'inverting RSA function without knowing the trapdoor'

$$y \equiv x^e \pmod{n}$$

✧ Factor the modulus \Rightarrow Break RSA

* If we can factor the modulus, we can break RSA

* If we can break RSA, we don't know whether we can factor the modulus... **open problem** (with negative evidences)

✧ Factor the modulus \Leftrightarrow Calculate private key d

* If we can factor the modulus, we can calculate the private exponent d (the trapdoor information).

* If we have the private exponent d , we can factor the modulus.

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Factoring reduces to RSA key recovery

- ✧ DeLaurentis, “A Further Weakness in the Common Modulus Protocol for the RSA Cryptosystem,” Cryptologia, Vol. 8, pp. 253-259, 1984
 - * If you have a pair of RSA public-key/private-key, you can factor $n=p \cdot q$ with a probabilistic algorithm.
 - * An example of the Universal Exponent Factorization method
- ✧ Basic idea: find a number b , $0 < b < n$ s.t.
 - $b^2 \equiv 1 \pmod{n}$ and $b \not\equiv \pm 1 \pmod{n}$ i.e. $1 < b < n-1$
 - * Note: There are four roots to the equation $b^2 \equiv 1 \pmod{n}$, ± 1 are two of them, all satisfy $(b+1)(b-1) = k \cdot n = k \cdot p \cdot q$, since $0 < b-1 < b+1 < n$, we have either $(p \mid b-1 \text{ and } q \mid b+1)$ or $(q \mid b-1 \text{ and } p \mid b+1)$, therefore, one of the factor can be found by $\gcd(b-1, n)$ and the other by $n/\gcd(b-1, n)$ or $\gcd(b+1, n)$

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Factoring reduces to RSA key recovery

- ✧ Algorithm to find b : $\Pr\{\text{success per repetition}\} = 1/2$
 1. Randomly choose a , $1 < a < n-1$, such that $\gcd(a, n) = 1$
 2. Find minimal j , $a^{2^j h} \equiv 1 \pmod{n}$ (where h satisfies $e \cdot d - 1 = 2^t h$)
 3. $b = a^{2^{j-1} h}$, if $b \not\equiv -1 \pmod{n}$, then $\gcd(b-1, n)$ is the result, else repeat 1-3
- ✧ Note: If we randomly choose $b \in \mathbb{Z}_n^*$ and find out that $b^2 \equiv 1 \pmod{n}$, the probability that $b=1$, $b=-1$, $b=c(\neq \pm 1)$, or $b=-c(\neq \pm 1)$ would be equal; $\Pr\{\text{success}\} = \Pr\{a^{2^{j-1} h} \not\equiv \pm 1\} = 1/2$
- ✧ Ex: $p=131$, $q=199$, $n=p \cdot q=26069$, $e=7$, $d=22063$
 - $\phi(n)=(p-1)(q-1)=25740=2^2 \cdot 6435 \mid ed-1=154440=2^3 \cdot 19305$,
 - choose $a=3$, try $j=1$ ($3^{2^1 \cdot 19305} \equiv 1$), $b = a^{2^{j-1} h} = 3^{19305} = 5372 (\neq \pm 1)$
 - $p = \gcd(b-1, n) = \gcd(5371, 26069) = 131$, $q = n/p = 199$

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Factoring reduces to RSA key recovery

- ✧ The above result says that “if you can recover a pair of RSA keys, you can factor the corresponding $n=p \cdot q$ ” i.e. “once a private key d is compromised, you need to choose a new pair of (n, e) instead of changing e only”
- ✧ The above result suggests that a scheme using (n, e_1) , (n, e_2) , ... (n, e_k) with a common n for each k participants without giving each one the value of p, q is insecure. You should not use the same n as some others even though you are not explicitly told the value of p and q .

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Factoring reduces to RSA key recovery

- ✧ The above result also suggests that if you can recover arbitrary RSA key pair, you can solve the problem of factoring n . Whenever you get an n , you can form an RSA system with some e (assuming $\gcd(e, \phi(n))=1$), then use your method to solve the private exponent d without knowing p and q , after that you can factor n .
- ✧ Although factoring is believed to be hard, and factoring breaks RSA, breaking RSA does not simplify factoring. Trivial non-factoring methods of breaking RSA could therefore exist. (What does it mean by breaking RSA? plaintext recovery? key recovery?...) different things

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Deterministic Encryption

- ✧ RSA Cryptosystem is a deterministic encryption scheme, i.e. a plaintext message is encrypted to a fixed ciphertext message
- ✧ Suffers from chosen plaintext attack
 - ★ an attacker compiles a large codebook which contains the ciphertexts corresponding to all possible plaintext messages
 - ★ in a two-message scheme, the attacker can always distinguish which plaintext was transmitted by observing the ciphertext (does not satisfy the Semantic Security Notation)
- ✧ Add randomness through padding

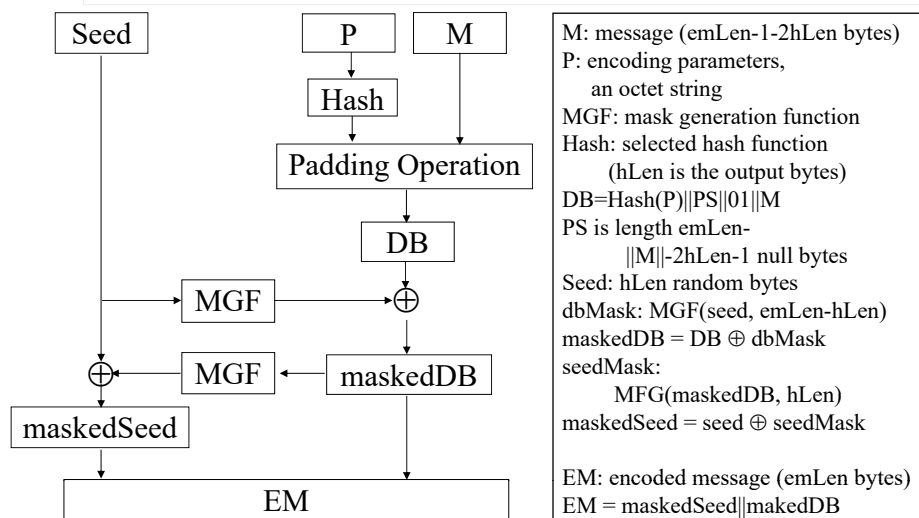
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RSA PKCS #1 v1.5 padding

- ✧ E.g. $k=128$ bytes (1024 bits) PKCS#1 v1.5 RSA
 - ★ plaintext message M (at most $128-3-8=117$ bytes)
 - ★ pseudorandom nonzero string PS (at least 8 bytes)
 - ★ message to be encrypted $m = 00||02||PS||00||M$
 - ★ encryption: $c \equiv m^e \pmod{n}$
 - ★ decryption: $m \equiv c^d \pmod{n}$
- ✧ c is now random corresponding to a fixed m , however, this only adds difficulties to the compilation of ciphertexts (a factor of 2^{64} times if PS is 8 bytes)

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PKCS #1 v2 padding - OAEP



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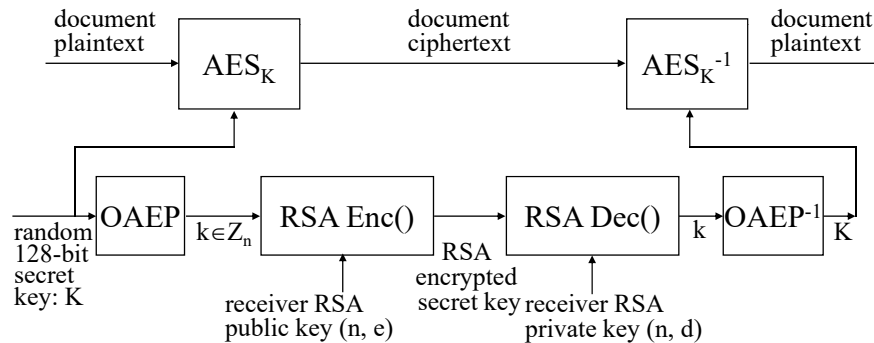
PKCS #1 v2 padding - OAEP

- ✧ Optimal Asymmetric Encryption (OAE)
 - ★ M. Bellare, "Optimal Asymmetric Encryption - How to Encrypt with RSA," Eurocrypt'94
- ✧ Optimal Padding in the sense that
 - ★ RSA-OAEP is semantically secure against adaptive chosen ciphertext attackers in the random oracle model
 - ★ the message size in a k -bit RSA block is as large as possible (make the most advantage of the bandwidth)
- ✧ Following by more efficient padding schemes:
 - ★ OAEP⁺, SAEP⁺, REACT

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Digital Envelop

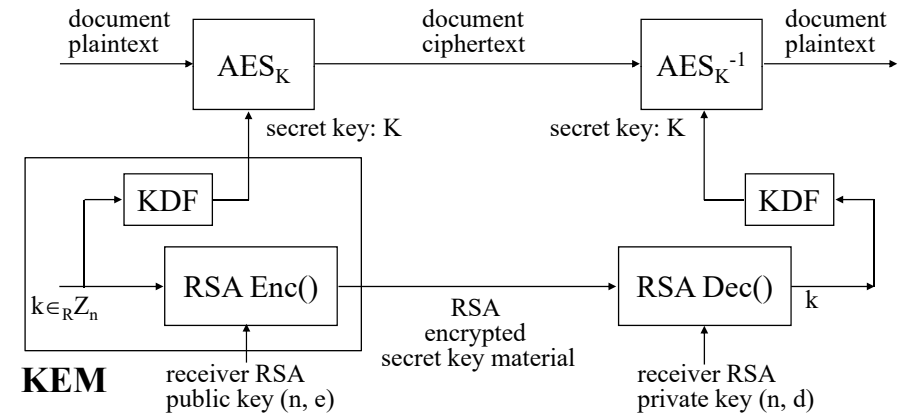
- Hybrid system (public key and secret key)
 - RSA is about 1000 times slower than AES
 - smaller exponent is faster (but more dangerous)



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KEM/DEM

- Key/Data Encapsulation Mechanism, hybrid scheme
- $k \xleftrightarrow{\text{OAEP}} K$, in a digital envelope scheme, K is a session key, might get compromised, forward security, requires OAEP



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RSA Fast Decryption with CRT

- Public key (n, e)

$n = p \cdot q$, p and q are large prime integers
 $\gcd(e, \phi(n)) = 1$ s.t. $\exists d, e \cdot d \equiv 1 \pmod{\phi(n)}$
 $\phi(n) = (p-1)(q-1)$ $3 \leq e \leq n-1$

- Private Key (n, d) or $(n, p, q, dp, dq, qInv)$

$e \cdot dp \equiv 1 \pmod{p-1}$
 $e \cdot dq \equiv 1 \pmod{q-1}$
 $q \cdot qInv \equiv 1 \pmod{p}$

- Encryption $c \equiv m^e \pmod{n}$

- Decryption $m \equiv c^d \pmod{n}$ or

$$\begin{aligned}
 & \left\{ \begin{array}{l} m_1 \equiv c^{dp} \pmod{p} \\ m_2 \equiv c^{dq} \pmod{q} \end{array} \right. \quad \left\{ \begin{array}{l} m_1 \equiv (m^e)^{dp} \equiv m^{e \cdot dp} \equiv m \pmod{p} \\ m_2 \equiv (m^e)^{dq} \equiv m^{e \cdot dq} \equiv m \pmod{q} \end{array} \right. \\
 & h \equiv qInv \cdot (m_1 - m_2) \pmod{p} \\
 & m \equiv m_2 + h \cdot q \pmod{n} \quad \left\{ \begin{array}{l} m \equiv m_2 \pmod{q} \text{ and} \\ m \equiv m_2 + qInv \cdot (m_1 - m_2) \cdot q \equiv m_1 \pmod{p} \end{array} \right.
 \end{aligned}$$

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Multi-Prime RSA

- RSA PKCS#1 v2.0 Amendment 1
- the modulus n may have more than two prime factors
- only private key operations and representations are affected $(p, q, dp, dq, qInv)$ (r_i, d_i, t_i)
 - $n = r_1 \cdot r_2 \cdot \dots \cdot r_k$, $k \geq 2$, where $r_1 = p$, $r_2 = q$
 - $e \cdot d_i \equiv 1 \pmod{r_i - 1}$, $i = 3, \dots, k$
 - $r_1 \cdot r_2 \cdot \dots \cdot r_{i-1} \cdot t_i \equiv 1 \pmod{r_i}$, $i = 3, \dots, k$

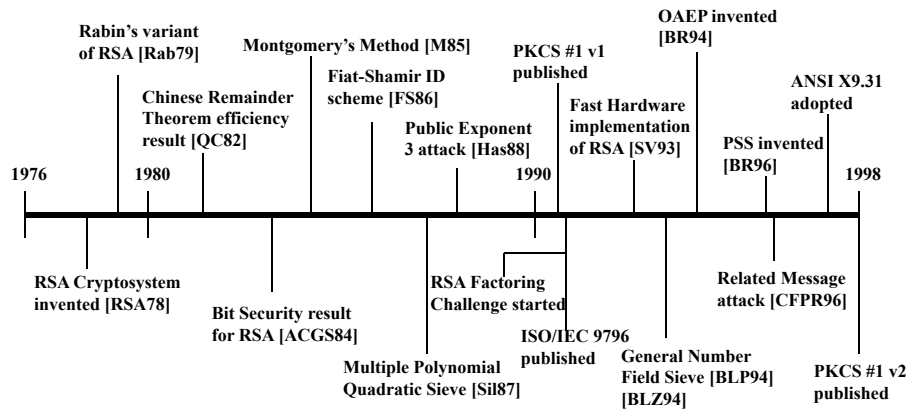
- Decryption:

- $m_1 \equiv c^{dp} \pmod{p}$
- $m_2 \equiv c^{dq} \pmod{q}$
- if $k > 2$ $m_i \equiv c^{d_i} \pmod{r_i}$, $i = 3, \dots, k$
- $h \equiv (m_1 - m_2) \cdot qInv \pmod{p}$
- $m = m_2 + q \cdot h$
- if $k > 2$, $R = r_1$, for $k = 3$ to k do
 - $R = R \cdot r_{i-1}$
 - $h \equiv (m_1 - m) \cdot t_i \pmod{r_i}$
 - $m = m + R \cdot h$

- advantages: lower computational cost for the decryption (and signature) primitives if CRT is used (also see 6.8.14)

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Factoring & RSA Timeline



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Alternative PKC's

- ✧ ElGamal Cryptosystem (Discrete-log based)
 - * Also suffers from long keys
- ✧ NTRU (Lattice based)
 - * Utilizes short keys
 - * Proprietary (License issues prevent from wide implementation)
 - * Recently, a weakness found in the signature scheme
- ✧ Elliptic Curve Cryptosystems
 - * Emerging public key cryptography standard for constrained devices.
- ✧ Paillier Cryptosystem (High order composite residue based)
- ✧ Goldwasser-Micali Cryptosystem (QR based)
 - * very low efficiency

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Miller-Rabin Primality Test

Why does it work?

bottom line of Miller-Rabin test

- ★ if n is prime, $a^{n-1} \equiv 1 \pmod{n}$ (Fermat Little theorem)
- ★ therefore, if $b_k \equiv a^{2^k m} \equiv a^{n-1} \equiv 1 \pmod{n}$, n must be composite
- ★ however, there are many composite numbers that satisfy $a^{n-1} \equiv 1 \pmod{n}$, Miller-Rabin test can detect many of them
- ★ $b_0, b_1, \dots, b_{k-1} (\equiv a^{(n-1)/2} \pmod{n})$ is a sequence s.t. $b_{i-1}^2 \equiv b_i \pmod{n}$
- ★ we consider only $b_{k-1}^2 \equiv a^{n-1} \equiv 1 \pmod{n}$ ← n is pseudo prime
- ★ if $b_i \equiv 1$ and $b_{i-1} \not\equiv \pm 1$, then n is composite ← basic factoring principle
- ★ if $b_i \equiv 1$ and $b_{i-1} \equiv 1$, consider b_{i-1} and then $b_{i-2} \dots$
- ★ if $b_0 \equiv 1$, could be prime, no guarantee
- ★ if $b_i \equiv 1$ and $b_{i-1} \equiv -1$ ($b_{i-2} \not\equiv \pm 1$), could be prime, no guarantee

there is no chance to apply basic factoring principle

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Miller-Rabin Primality Test

In summary:

$b_0, b_1, b_2, \dots, b_{i-1}, b_i, \dots, b_k$

there are four cases:

- ★ Case 1: $b_k \neq 1$ n is a composite number
- ★ Case 2: $b_k = 1$, let i be the minimal $i, k \geq i > 0$ such that $b_i = 1$ and $b_{i-1} \neq \pm 1$ n is a composite number (with nontrivial factors calculated)
- ★ Case 3: $b_k = 1$, let i be the minimal $i, k \geq i > 0$ such that $b_i = 1$ and $b_{i-1} = -1$ a pseudo prime number
- ★ Case 4: $b_k = 1, b_0 = 1$ a pseudo prime number

4 possible sequences for $b_0, b_1, b_2, \dots, b_{i-1}, b_i, \dots, b_k$:

342, 22, 5, 1, 1, 1, ..., 1	composite, factored
45, 5634, 325, 213, -1, 1, ..., 1	possibly prime
1, 1, 1, ..., 1	possibly prime
214, 987, ..., 8931, 321, 134	composite

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M-R Test: Prime Modulus

- ◇ consider n being a *prime number* p
- ◇ $p-1$ is an even number, therefore, let $p-1=2^k \cdot m$, m is odd
- ◇ choose one $a \in \mathbb{Z}_p^*$, let r be the smallest integer s.t. $a^r \equiv 1 \pmod{p}$, i.e. r is the order of a modulo p , $\text{ord}_p(a)$
- ◇ (exercise 3.9) $a^{p-1} \equiv 1 \pmod{p} \Rightarrow r \mid p-1$
- ◇ because $r \mid p-1 (= 2^k \cdot m)$, one of $\{m, 2 \cdot m, 2^2 \cdot m, \dots, 2^k \cdot m\}$ might be r (probability reduces if m has many factors)
- ◇ Case 1: if “ $2^i \cdot m$ (for some $i > 0$) is r ”, $a^{2^{i-1} \cdot m}$ must be -1
 - ★ r is the smallest integer s.t. $a^r \equiv 1 \Rightarrow$ square root of a^r must be -1
 - ★ $\{a^m, a^{2 \cdot m}, \dots, a^{2^i \cdot m}\}$ is $\{?, ?, -1, 1, \dots, 1\}$
- ◇ Case 2: if “none of $2^i \cdot m$ is r ” or “ m is r ”, $a^{2^i \cdot m}$ must all be 1 ,
 - ★ $\{a^m, a^{2 \cdot m}, \dots, a^{2^i \cdot m}\}$ is $\{1, 1, 1, 1, \dots, 1\}$
 - ★ try some other $a \in \mathbb{Z}_p^*$

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Miller-Rabin Primality Test

Why does it work??? an inside view

- ✧ $b_i \equiv 1 \pmod{n}$ and $b_{i-1} \not\equiv \pm 1 \pmod{n}$ happens when $b_i \equiv 1 \pmod{p_i}$ for all prime factors p_i of n and

$$\begin{array}{l} b_{i-1} \equiv 1 \pmod{p_i} \text{ for some prime factors } p_i \text{ but} \\ b_{i-1} \equiv -1 \pmod{q_i} \text{ for other prime factors } q_i \end{array}$$

Note: for a prime modulus p , $a^{\text{ord}_p(a)} \equiv 1 \pmod{p}$
if $\text{ord}_p(a)$ is even then $a^{\text{ord}_p(a)/2} \equiv -1 \pmod{p}$

- ✧ e.g. $n = 561 = 3 \times 11 \times 17$, $560 = 16 \times 35 = 2^4 \times 35$
let $a = 2$

$$b_0 = 263 \pmod{561} \equiv -1 \pmod{3} \equiv -1 \pmod{11} \equiv 8 \pmod{17}$$

$$b_1 = 166 \pmod{561} \equiv 1 \pmod{3} \equiv 1 \pmod{11} \equiv -4 \pmod{17}$$

$$b_2 = 67 \pmod{561} \equiv 1 \pmod{3} \equiv 1 \pmod{11} \equiv -1 \pmod{17}$$

$$b_3 \equiv 1 \pmod{561} \equiv 1 \pmod{3} \equiv 1 \pmod{11} \equiv 1 \pmod{17}$$

i.e. inconsistent progress w.r.t each prime factor

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Subset Sum Problem is NP-Complete

- ✧ Subset Sum Problem (SSP)

Given a set B of positive numbers and a number d

- * Search SSP: find a subset $\{b_j\} \subseteq B$ s.t. $d = \sum b_j$
- * Decision SSP: decide if there exists a subset $\{b_j\} \subseteq B$ s.t. $d = \sum b_j$
- * Decision SSP is equivalent to Search SSP: (by elimination)

- ✧ Subset Sum Problem is NP-complete

- * Cook-Levin Thm: Satisfiability Problem (SAT) is NP-Complete
- * $\text{SAT} \leq_M \text{SSP}$: there exists a poly-time reduction to convert a formula ϕ to an instance $\langle B, d \rangle$ of SSP problem
 - ✧ If the formula ϕ is satisfiable, $\langle B, d \rangle \in \text{SSP}$
 - ✧ If $\langle B, d \rangle \in \text{SSP}$, formula ϕ is satisfiable

Therefore, SSP is also NP-complete

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$\text{SAT} \leq_M \text{D-Subset Sum}$

- ✧ Given a formula ϕ with k clauses C_1, C_2, \dots, C_k and n variables

- * For each variable x , create 2 integers n_{xt} and n_{xf}
- * For each clause C_j of length ℓ_j , create $\ell_j - 1$ integers m_{j1}, m_{j2}, \dots
- * Choose t so that T must contain exactly one of each $(n_{xt}$ or $n_{xf})$ pairs and at least one from each clause

- ✧ This construction can be carried out in poly-time

- ✧ ϕ is satisfiable iff there exists solution to this SSP

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$\text{SAT} \leq_M \text{D-Subset Sum (cont'd)}$

Example: $(x \vee y \vee z) \wedge (\neg x \vee \neg a) \wedge (a \vee b \vee \neg y \vee \neg z)$

	x	y	z	a	b	C_1	C_2	C_3
n_{xt}	1	0	0	0	0	1	0	0
n_{xf}	1	0	0	0	0	0	1	0
n_{yt}		1	0	0	0	1	0	0
n_{yf}		1	0	0	0	0	0	1
n_{zt}			1	0	0	1	0	0
n_{zf}			1	0	0	0	0	1
n_{at}				1	0	0	0	1
n_{af}				1	0	0	1	0
n_{bt}					1	0	0	1
n_{bf}					1	0	0	0
m_{11}						1	0	0
m_{12}						1	0	0
m_{21}						0	1	0
m_{31}						0	0	1
m_{32}						0	0	1
m_{33}						0	0	1
t	1	1	1	1	1	3	2	4

Encode all numbers with a base larger than all entries of t e.g. 10

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